LPENSL-TH-03/08

Correlation functions of the open XXZ chain II

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Abstract

We derive compact multiple integral formulas for several physical spin correlation functions in the semi-infinite XXZ chain with a longitudinal boundary magnetic field. Our formulas follow from several effective re-summations of the multiple integral representation for the elementary blocks obtained in our previous article (I). In the free fermion point we compute the local magnetization as well as the density of energy profiles. These quantities, in addition to their bulk behavior, exhibit Friedel type oscillations induced by the boundary; their amplitudes depend on the boundary magnetic field and decay algebraically in terms of the distance to the boundary.

1 Introduction

The Hamiltonian of the Heisenberg XXZ spin-1/2 finite chain [1] with diagonal boundary conditions (namely with longitudinal boundary magnetic fields) is defined as [2,3]

$$\mathcal{H} = \sum_{m=1}^{M-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right\} + h_- \sigma_1^z + h_+ \sigma_M^z. \tag{1.1}$$

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This is a linear operator acting in the quantum space $\mathcal{H} = \bigotimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \simeq \mathbb{C}^2$, of dimension 2^M of the chain. In this expression, $\sigma_m^{\pm}$, σ_m^z denote local spin operators (acting as Pauli matrices) at site m, Δ is the anisotropy parameter and $h_{\pm}$ are the boundary (longitudinal) magnetic fields.

We have recently developed a method to compute the so-called elementary blocks of correlation functions for this model (see [4], that we refer to as Paper I in the following) in the framework of the (algebraic) Bethe ansatz [5–18] for boundary integrable systems [2, 3, 19–31]. The results essentially agree with previous expressions derived from the vertex operator approach [32, 33]. The purpose of the present paper is to obtain the physical spin correlation functions for this model, in particular, the one point functions for the local spin operators at distance m from the boundary as well as several two point functions (like boundary-bulk correlation functions). There are numerous physical interests in such quantities that can be measured in actual experiments [34–46].

In much the same way as in the bulk case [47–52], the computation of the physical correlation functions amounts to obtain effective re-summations of the multiple integral representations derived for the elementary blocks. For example, the one point functions at distance m from the boundary, such as the local magnetization $\langle \sigma_m^z \rangle$, can be written as a sum of 2^m elementary blocks. We will show how to obtain compact expressions for such objects, typically involving the sum of only m terms, each containing multiple integrals whose integrants have a structure similar to the one of the elementary blocks. In the free fermion point we are able to compute these multiple integrals (and hence the corresponding correlations functions) almost completely by reducing them to single integrals. For instance the local magnetization and the density of energy profiles (a quantity of interest in the study and the understanding of the interplay between quantum entanglement and quantum criticality [53–61]) are expressed as single integrals. Hence, their asymptotic behavior at long distance m from the boundary can be explicitly evaluated. In addition to the bulk constant value they exhibit Friedel type oscillations [44–46, 60, 61], algebraically decaying with the distance m, their amplitudes being rational functions of the boundary magnetic field, in agreement with field theory predictions [38–42, 62–70].

We start this paper with a short technical introduction concerning the algebraic Bethe ansatz approach to the open XXZ spin-1/2 chain subject to diagonal boundary magnetic fields. This preliminary section is followed in Section 3 by a reminder of the method proposed in [4] to compute correlation functions of open integrable models in the framework of algebraic Bethe ansatz. In Section 4 we obtain formulae for the action of local operators on arbitrary boundary states in a form suitable for taking later on the thermodynamic limit. Using these results, we derive a series representation for the generating function $\langle \mathcal{Q}_m(\kappa) \rangle$ of bulk-boundary σ^z correlation functions in Section 5. This formula is the boundary analogue of the original series [49] in the bulk case. In Section 6, we obtain a formula for $\langle \mathcal{Q}_m(\kappa) \rangle$ alternative to the one inferred in Section 5. We also give multiple integral representations for $\langle \sigma_{m+1}^{+} \sigma_1^{1} \rangle$ and for the local density of energy. These formulae are obtained by a direct resummation

of the corresponding elementary blocks. It is worth stressing that we actually have two representations for their integrand. The first one is in the spirit of the bulk case [52] and involves the Izergin determinant representation [71] for the partition function of the six vertex model with domain wall boundary conditions. The second one involves the Tsuchiya [72] determinant representation for the partition function of the six vertex model with reflecting ends. The next section is devoted to the free fermion point. For that case are able to reduce the multiple integrals to one dimensional ones. This allows us to write the leading asymptotics of the local magnetization and of the density of energy profiles as well as of the $\langle\sigma_{m+1}^{+}\sigma_{1}^{-}\rangle$ correlation function. Our conclusions are presented in the last section.

2 The open XXZ spin-$1/2$ chain

The spectrum of $\mathcal{H}$ can be obtained by algebraic Bethe ansatz (ABA) [3]. The central tool of this method is the boundary monodromy matrix, which will be defined after we introduce some necessary notations.

Here and in the following we adopt the standard parameterizations $\Delta = \cosh\eta$ and $h_{\pm} = \sinh\eta\coth\xi_{\pm}$.

Let $R : \mathbb{C} \to \mathrm{End}(V \otimes V)$, $V \simeq \mathbb{C}^2$, be the R-matrix of the six-vertex model, obtained as the trigonometric solution of the Yang-Baxter equation:

$$R(u) = \sinh(u+\eta)\ \widehat{R}(u), \qquad \text{with} \quad \widehat{R}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

and

$$b(u) = \frac{\sinh u}{\sinh(u+\eta)}, \quad c(u) = \frac{\sinh\eta}{\sinh(u+\eta)}. \tag{2.2}$$

The bulk monodromy matrix $T(\lambda) \in \mathrm{End}(V_0 \otimes \mathcal{H})$, $V_0 \simeq \mathbb{C}^2$, is defined as an ordered product of R matrices:

$$T_0(\lambda) = R_{0M}(\lambda - \xi_M) \dots R_{01}(\lambda - \xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]}. \tag{2.3}$$

The subscript 0 labels here the two-dimensional auxiliary space V_0, whereas subscripts m running from 1 to M refer to the quantum spaces $\mathcal{H}_m$ of the chain. Besides, we attach an inhomogeneity parameter ξ_m to each site m of the chain. We recall that $T(\lambda)$ satisfies the Yang-Baxter algebra, on $V_0 \otimes V_{0'} \otimes \mathcal{H}$:

$$R_{00'}(\lambda-\mu)\, T_0(\lambda)\, T_{0'}(\mu) = T_0(\lambda)\, T_{0'}(\mu)\, R_{00'}(\lambda-\mu). \tag{2.4}$$

Let us also introduce the two boundary matrices, $K_{\pm}(\lambda) = K(\lambda \pm \eta/2; \xi_{\pm})$, where $K(\lambda;\xi)$ is the 2×2 matrix acting on the auxiliary space:

$$K(\lambda;\xi) = \begin{pmatrix} \sinh(\lambda+\xi) & 0 \\ 0 & \sinh(\xi-\lambda) \end{pmatrix}_{[0]}. \tag{2.5}$$

The boundary monodromy matrix $\mathcal{U}(\lambda)$ [3] is built out of a product of $T(\lambda)$ and $K_+(\lambda)$, namely[1]

$$\mathcal{U}_0^{t_0} = T_0^{t_0}(\lambda)\, K_+^{t_0}(\lambda)\, \widehat{T}_0^{t_0}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}_{[0]}^{t_0} , \tag{2.6}$$

where

$$\begin{aligned} \widehat{T}_0(\lambda) &= R_{10}(\lambda+\xi_1-\eta)\dots R_{M0}(\lambda+\xi_M-\eta) \\ &= (-1)^M \prod_{j=1}^{M} [\sinh(\lambda+\xi_j)\sinh(\lambda+\xi_j-\eta)]\;\; T_0^{-1}(-\lambda+\eta) \;\; . \end{aligned} \tag{2.7}$$

This boundary monodromy matrix satisfies the reflection algebra first introduced in [20]:

$$\begin{aligned} &R_{00'}(-\lambda+\mu)\,\mathcal{U}_0^{t_0}(\lambda)\;R_{00'}(-\lambda-\mu-\eta)\,\mathcal{U}_{0'}^{t_{0'}}(\mu) \\ &\qquad = \mathcal{U}_{0'}^{t_{0'}}(\mu)\;R_{00'}(-\lambda-\mu-\eta)\,\mathcal{U}_0^{t_0}(\lambda)\;R_{00'}(-\lambda+\mu)\,. \end{aligned} \tag{2.8}$$

The commuting charges of the XXZ spin-1/2 chain with diagonal boundary conditions are realized by the one-parameter family of transfer matrices:

$$\mathcal{T}(\lambda) = \mathrm{tr}_0\,[\,\mathcal{U}_0(\lambda)\,K_-(\lambda)]\;\; , \tag{2.9}$$

and the Hamiltonian (1.1) is obtained in terms of the derivative $\dfrac{\mathrm{d}\mathcal{T}(\lambda)}{\mathrm{d}\lambda}\,|_{\lambda=\eta/2}$ in the homogeneous case $\xi_i = \eta/2$, $i = 1,\dots,M$.

Common eigenstates of all transfer matrices (and thus of the Hamiltonian (1.1) in the homogeneous case) can be constructed by successive actions of $\mathcal{B}(\lambda)$ operators on the reference state $|0\,\rangle$ which is the ferromagnetic state with all the spins up. More precisely, the state [2]

$$|\,\{\lambda\}_1^n\,\rangle_b \equiv \mathcal{B}(\lambda_1)\dots\mathcal{B}(\lambda_n)\,|0\,\rangle \tag{2.10}$$

is a common eigenstate of the transfer matrices if the set of spectral parameters $\{\lambda\}_1^n \equiv \{\lambda_j\}_{1\le j\le n}$ is a solution of the Bethe equations

$$y_j(\lambda_j;\{\lambda\}_1^n) = y_j(-\lambda_j;\{\lambda\}_1^n)\,, \qquad j=1,\dots,n, \tag{2.11}$$

where

$$\begin{aligned} y_j(x;\{\lambda\}_1^n) &= \frac{\hat{y}(x;\{\lambda\}_1^n)}{\mathfrak{s}(\lambda_j, x-\eta)}\;, \\ \hat{y}(x;\{\lambda\}_1^n) &= -a(x)\;d(-x)\sinh(x+\xi_+-\eta/2)\sinh(x+\xi_--\eta/2) \\ &\qquad\qquad \times \prod_{l=1}^{n} \mathfrak{s}(x-\eta,\lambda_l) \;\; . \end{aligned} \tag{2.12}$$

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[1] Note that it corresponds to the matrix $\mathcal{U}_+$ of our previous article (I). Since we consider only the '+' case in the present article, we do not specify it in the notations.

[2] In order to lighten the formulae we have slightly changed the notation with respect to the one in Paper I. Namely, the vector $|\,\{\lambda\}_1^n\,\rangle_b$ corresponds to $|\psi_+(\{\lambda\})\,\rangle$ in [4]. Such a boundary state should in particular be distinguished from the corresponding bulk state that we merely denote $|\,\{\lambda\}_1^n\,\rangle$.

Here and in the following, $\mathfrak{s}(\lambda,\mu)$ denotes the function

$$\mathfrak{s}(\lambda,\mu)=\sinh(\lambda+\mu)\sinh(\lambda-\mu)\ , \tag{2.13}$$

and the functions $a(\lambda)$ and $d(\lambda)$ stand respectively for the eigenvalues of the bulk operators $A(\lambda)$ and $D(\lambda)$ on the pseudo-vacuum $|0\rangle$:

$$a(\lambda)=\prod_{i=1}^{M}\sinh(\lambda-\xi_i+\eta)\,,\qquad d(\lambda)=\prod_{i=1}^{M}\sinh(\lambda-\xi_i)\,. \tag{2.14}$$

Of course it is also possible to implement the Bethe ansatz starting from the dual state $\langle 0|$ and acting on it with $\mathcal{C}(\lambda)$ operators:

$$_b\langle\{\lambda\}_1^n\,|\equiv\langle 0\,|\mathcal{C}(\lambda_1)\ldots\mathcal{C}(\lambda_n)\,. \tag{2.15}$$

The description of the ground state of $\mathcal{H}$ in the half-infinite chain depends on the regime. One should distinguish the two domains $-1<\Delta\le 1$ (massless regime) and $\Delta>1$ (massive regime):

$$\begin{aligned}
&\alpha_j=\lambda_j,\quad \zeta=i\eta>0,\quad \xi_-=-i\tilde{\xi}_-,\quad \text{with } -\frac{\pi}{2}<\tilde{\xi}_-\le\frac{\pi}{2},\qquad \text{for } -1<\Delta\le 1,\\
&\alpha_j=i\lambda_j,\quad \zeta=-\eta>0,\quad \xi_-=-\tilde{\xi}_-+i\delta\frac{\pi}{2},\quad \text{with } \tilde{\xi}_-\in\mathbb{R},\qquad \text{for } \Delta>1,
\end{aligned}$$

where $\delta=1$ for $|h_-|<\sinh\zeta$ and $\delta=0$ otherwise. Thus, to a given set of roots $\{\lambda_j\}$ corresponds a set of variables $\{\alpha_j\}$ given by the previous change of variables. Note that the nature of the ground state rapidities depends on the value of the boundary field h_-.

Indeed, when $\tilde{\xi}_-<0$ or $\tilde{\xi}_->\zeta/2$, the ground state of the Hamiltonian (1.1) is given in both regimes by the maximum number N of roots λ_j corresponding to real (positive) α_j such that $\cos p(\lambda_j)<\Delta$. In the thermodynamic limit $M\to\infty$, these roots λ_j form a dense distribution on an interval $[0,\Lambda]$ of the real or imaginary axis. Their density

$$\rho(\lambda_j)=\lim_{M\to\infty}[M(\lambda_{j+1}-\lambda_j)]^{-1} \tag{2.16}$$

satisfies the integral equation

$$2\pi\rho(\lambda)+\int_{-\Lambda}^{\Lambda}\frac{i\sinh(2\eta)}{\mathfrak{s}(\lambda-\mu,\eta)}\rho(\lambda)\,\mathrm{d}\lambda=\frac{2i\sinh\eta}{\mathfrak{s}(\lambda,\eta/2)}, \tag{2.17}$$

with $\Lambda=+\infty$ in the massless regime, and $\Lambda=-i\pi/2$ in the massive one. The density can be expressed in terms of usual functions:

$$\rho(\lambda)=\begin{cases}\dfrac{1}{\zeta\cosh(\pi\lambda/\zeta)}, & -1<\Delta<1;\\[2mm] \dfrac{i}{\pi}\displaystyle\prod_{n\ge 1}\left(\frac{\sinh n\zeta}{\cosh n\zeta}\right)^2\frac{\theta_3(i\lambda;-\zeta)}{\theta_4(i\lambda;-\zeta)}, & 1<\Delta.\end{cases} \tag{2.18}$$

However, when $0 < \tilde{\xi}_- < \zeta/2$, the ground state also admits a root $\check{\lambda}$ (corresponding to a complex $\check{\alpha}$) which tends to $\eta/2 - \xi_-$ with exponentially small corrections in the large M limit. In that case, the density of real roots is still given by the solution of (2.17).

3 The ABA approach to correlation functions

A zero temperature correlation function is the normalized expectation value, in the ground state of the Hamiltonian (1.1), of some *local*[3] operator $\mathcal{O}_m$,

$$\langle \mathcal{O}_m \rangle = \frac{{}_b\langle \{\lambda\}_1^N \, |\mathcal{O}_m| \, \{\lambda\}_1^N \rangle_b}{{}_b\langle \{\lambda\}_1^N \, | \, \{\lambda\}_1^N \rangle_b}, \tag{3.1}$$

where the parameters λ are the solutions of the ground state Bethe equations.

In order to compute such a correlation function, one should first derive the action of the corresponding local operator on the boundary state $| \, \{\lambda\}_1^N \rangle_b$, and then evaluate the resulting scalar products. We have constructed in [4] a method to solve this problem. This method is based on a revisited version of the quantum inverse problem, first introduced in [47, 73] for the XXZ spin chain with periodic boundary conditions. Once a local operator is reconstructed in terms of the entries of the bulk monodromy matrix, its action on boundary states can then be computed thanks to the decomposition of boundary states in terms of bulk states and to the Yang-Baxter commutation relations.

We shall now recall the main points of our method.

3.1 The bulk inverse problem revisited

Proposition 3.1 (Solution of the bulk inverse problem) *[47, 73] Let E_m^{ij} be an elementary matrix acting non-trivially only on the m^{th} site of the chain, then*

$$E_m^{ij} = \prod_{k=1}^{m-1} (A+D)(\xi_k) \;\mathrm{tr}\left(T_0(\xi_m) \, E_0^{ij} \right) \prod_{k=1}^{m} (A+D)^{-1}(\xi_k) \,. \tag{3.2}$$

Note that, thanks to the crossing symmetry of the R-matrix, one can recast the inverse of the bulk transfer matrix at inhomogeneity parameter $(A+D)^{-1}(\xi_k)$ in terms of the transfer matrix at shifted parameter $(A+D)(\xi_k - \eta)$, namely

$$(A+D)^{-1}(\xi_k) = \frac{(A+D)(\xi_k - \eta)}{a(\xi_k)\, d(\xi_k - \eta)} \;. \tag{3.3}$$

It is worth pointing out that the products of elementary matrices on the first m sites of the chain define a basis in the space of local operators $\mathcal{O}_m$, so that (3.2) allows one to define a reconstruction for all such operators. However, this reconstruction is

[3] *i.e.* acting non-trivially only in $\bigotimes_{k=1}^{m} \mathcal{H}_k$.

especially convenient when one wants to obtain the action on a bulk Bethe state; indeed, in such a case, the product of bulk transfer matrices merely produces a numerical factor. This is no longer the case when one acts on a boundary Bethe state. The theorem below allows one to reconstruct local operators in a way adapted to an action on boundary states.

Theorem 3.1 *[4] For any set of inhomogeneity parameters $\{\xi_{i_1},\dots,\xi_{i_n}\}$, the product of bulk operators*

$$T_{\epsilon_{i_n}\,\epsilon'_{i_n}}(\xi_{i_n})\dots T_{\epsilon_{i_1}\,\epsilon'_{i_1}}(\xi_{i_1})\,T_{\bar\epsilon_{i_1}\,\bar\epsilon'_{i_1}}(\xi_{i_1}-\eta)\dots T_{\bar\epsilon_{i_n}\,\bar\epsilon'_{i_n}}(\xi_{i_n}-\eta) \tag{3.4}$$

vanishes if, for some $k\in\{i_1,\dots,i_n\}$, $\epsilon_k=\bar\epsilon_k$.

Thus we have :

Corollary 3.1 *A product of elementary matrices acting on the first m sites of the chain can be expressed as a single monomial in the entries of the bulk monodromy matrix:*

$$\begin{aligned} E_1^{\epsilon_1\,\epsilon'_1}\dots E_m^{\epsilon_m\,\epsilon'_m} = &\prod_{i=1}^{m}\left[a(\xi_i)\,d(\xi_i-\eta)\right]^{-1} \\ &\times T_{\epsilon'_1\,\epsilon_1}(\xi_1)\dots T_{\epsilon'_m\,\epsilon_m}(\xi_m)\,T_{\bar\epsilon_m\,\bar\epsilon_m}(\xi_m-\eta)\dots T_{\bar\epsilon_1\,\bar\epsilon_1}(\xi_1-\eta) \end{aligned} \tag{3.5}$$

with $\bar\epsilon_i=\epsilon'_i+1 \pmod 2$.

This result represents a strong simplification. Indeed, it means that, over the 2^m monomials appearing in the reconstruction of a local operator (3.2), only one is non-zero. We shall now explain how to compute the action of this non-vanishing monomial on an arbitrary (bulk or boundary) state.

3.2 Action on bulk and boundary states

Before stating the lemma which explains how to derive the action of the former monomial on a bulk state, we recall that the action of $A(\mu)$ or $D(\mu)$ on a bulk state $|\,\{\lambda\}_1^N\,\rangle\equiv\prod_{j=1}^{N}B(\lambda_j)|0\,\rangle$ produces two kinds of terms: the *direct term*, where all rapidities remain unchanged, and *indirect terms* where one λ_j is replaced by μ.

Lemma 1 (Action on a bulk state) *[4] The action on a bulk state $|\,\{\lambda\}_1^N\,\rangle$ of a string of operators*

$$\mathcal{O}^{\epsilon'_{i_1},\dots,\epsilon'_{i_n}}_{\epsilon_{i_1},\dots,\epsilon_{i_n}} = \underbrace{T_{\epsilon'_{i_n}\,\epsilon_{i_n}}(\xi_{i_n})\dots T_{\epsilon'_{i_1}\,\epsilon_{i_1}}(\xi_{i_1})}_{(1)}\,\underbrace{T_{\bar\epsilon_{i_1}\,\bar\epsilon_{i_1}}(\xi_{i_1}-\eta)\dots T_{\bar\epsilon_{i_n}\,\bar\epsilon_{i_n}}(\xi_{i_n}-\eta)}_{(2)} \tag{3.6}$$

with $\bar\epsilon_l=\epsilon'_l+1 \pmod 2$, satisfies the restrictions:

- *The only non-zero contributions of the tail operators (2) come from*

(i) the indirect action of all $A(\xi_l - \eta)$ operators;

(ii) the direct action of all $D(\xi_l - \eta)$ operators.

- *In what concerns the head operators* (1),

(iii) if $\epsilon'_l = 1$, the action of the operator $T_{\epsilon'_l \epsilon_l}(\xi_l)$ (i.e. $A(\xi_l)$ or $B(\xi_l)$) does not result in any substitution of a parameter $\xi_i - \eta$;

(iv) if $\epsilon'_l = 2$, the action of the operator $T_{\epsilon'_l \epsilon_l}(\xi_l)$ (i.e. $D(\xi_l)$ or $C(\xi_l)$) substitutes $\xi_l - \eta$ with ξ_l; moreover, if there were others parameters $\xi_j - \eta$, $j \neq l$, in the initial state, they are still present in the resulting state.

This lemma enables us to compute the action of local operators on any (arbitrary) bulk state. In order to compute the action on a boundary state, we use the fact that the latter can be decomposed in terms of bulk states:

Proposition 3.2 (Boundary-bulk decomposition) *[4], [74] Let $|\,\{\lambda\}_1^n\,\rangle_b$ be an arbitrary boundary state, then it can be expressed in terms of bulk states as*

$$|\,\{\lambda\}_1^n\,\rangle_b = \sum_{\substack{\sigma_i=\pm \\ i=1,\dots,n}} H^{\mathcal{B}}_{\{\sigma_i\}}\,(\{\lambda\}_1^n)\,|\,\{\lambda^\sigma\}_1^n\,\rangle, \tag{3.7}$$

with

$$H^{\mathcal{B}}_{\{\sigma_i\}}\,(\{\lambda\}_1^n) = \prod_{j=1}^n H^{\mathcal{B}}_{\sigma_j}(\lambda_j)\;\cdot \prod_{1\le r<s\le n} \frac{\sinh(\overline{\lambda}^\sigma_{rs} - \eta)}{\sinh(\overline{\lambda}^\sigma_{rs})}. \tag{3.8}$$

In this expression, $H^{\mathcal{B}}_\sigma(\lambda)$ denotes the "one-particle" boundary-bulk coefficient, which can be written as

$$H^{\mathcal{B}}_\sigma(\lambda) = \sigma\,(-1)^M d(-\lambda^\sigma)\,\frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda}\,\sinh(\lambda^\sigma + \xi_+ - \eta/2)\,. \tag{3.9}$$

Here we have used the notations:

$$\lambda_{rs} = \lambda_r - \lambda_s, \qquad \overline{\lambda}_{rs} = \lambda_r + \lambda_s, \tag{3.10}$$

$$\lambda_j^{\sigma_j} = \sigma_j\lambda_j, \quad \text{and more generally} \quad \{\lambda^\sigma\}_1^n = \left\{\lambda_j^{\sigma_j}\right\}_1^n\,. \tag{3.11}$$

It is remarkable that, by using this decomposition and the previous lemma, we are able to express the action of a local operator $\mathcal{O}_m$ on an arbitrary boundary state as a linear combination of such boundary states:

$$\mathcal{O}_m|\,\{\lambda\}_1^N\,\rangle_b = \sum_{\alpha_m} C_{\alpha_m}(\{\lambda\}\,;\{\xi\})\,|\,\{\mu_i\}_{i\in\alpha_m}\,\rangle_b, \tag{3.12}$$

where the summation is taken over certain subsets $\{\mu_i\}_{i\in\alpha_m}$ of $\{\lambda\}_1^N \cup \{\xi\}_1^m$, and where C_{α_m} are coefficients which can be computed generically[4].

[4] See section 5.3 of [4] for the explicit expression in the case of a product of elementary matrices.

3.3 From scalar products to correlation functions

It now remains, in order to obtain the correlation function (3.1), to take the scalar product of this resulting combination of states with the ground state ${}_b\langle\{\lambda\}_1^N\ |$. This can be done by using the trigonometric generalization [4] of the rational [74] formula for the scalar product between a boundary Bethe state and an arbitrary boundary state. In particular, we have to evaluate the following type of renormalized scalar product:

$$\mathbb{S}\left(\{\lambda\}\,,\{\mu\}\right)=\frac{{}_b\langle\{\lambda\}\,|\,\{\mu\}\,\rangle_b}{{}_b\langle\{\lambda\}\,|\,\{\lambda\}\,\rangle_b}, \tag{3.13}$$

where the sets $\{\lambda\}$ and $\{\mu\}$ are partitioned according to:

$$\{\lambda\}=\{\lambda_a\}_{a\in\alpha_-}\cup\{\lambda_b\}_{b\in\alpha_+}\ ,\qquad \{\mu\}=\{\lambda_a\}_{a\in\alpha_-}\cup\{\xi_b\}_{b\in\gamma_+}\ , \tag{3.14}$$

with $|\alpha_+|=|\gamma_+|$. Here the parameters λ are the solutions of the ground state boundary Bethe equations, $\{\xi_b\}_{b\in\gamma_+}$ are arbitrary inhomogeneities and $\alpha_+\cup\alpha_-$ is a partition of $\{1,\ldots,N\}$.

Since we are especially interested in the thermodynamic limit $M\to+\infty$ of the correlation function (3.1), we only recall the explicit formula for the leading asymptotic contribution of the renormalized scalar product [4]:

$$\mathbb{S}\left(\{\lambda\}\,,\{\mu\}\right)=\left(\frac{1}{M}\right)^{|\alpha_+|}\mathcal{S}(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\{\lambda\}_{\alpha_-})\ \det_{\substack{a\in\alpha_+\\ b\in\gamma_+}}\big[\widetilde{\mathcal{S}}_{ab}\big]. \tag{3.15}$$

The coefficient $\mathcal{S}(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\{\lambda\}_{\alpha_-})$ has been computed in [4]. Note that, since $\{\lambda\}$ is a solution of the boundary Bethe equations, there is some sign arbitrariness in the expression of this coefficient: indeed, it is in fact equal to

$$\begin{aligned}\mathcal{S}_\sigma(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\{\lambda\}_{\alpha_-}) &= \frac{\prod\limits_{\substack{a,b\in\alpha_+\\ a>b}}\mathfrak{s}\left(\lambda_b,\lambda_a\right)}{\prod\limits_{\substack{a,b\in\gamma_+\\ a>b}}\mathfrak{s}\left(\xi_b,\xi_a\right)}\prod_{a\in\alpha_-}\frac{\prod\limits_{b\in\alpha_+}\mathfrak{s}\left(\lambda_b,\lambda_a\right)}{\prod\limits_{b\in\gamma_+}\mathfrak{s}\left(\xi_b,\lambda_a\right)}\\ &\times\prod_{b\in\gamma_+}\frac{\hat{y}(\xi_b;\{\lambda\}_{\alpha_+\cup\alpha_-})\sinh(2\xi_b+\eta)}{\sinh(2\xi_b)}\prod_{a\in\alpha_+}\frac{\sinh(2\lambda_a^\sigma-\eta)\sinh(2\lambda_a)}{\hat{y}(\lambda_a^\sigma;\{\lambda\}_{\alpha_+\cup\alpha_-})\sinh(2\lambda_a+\eta)},\end{aligned} \tag{3.16}$$

for any value of $\sigma_a\in\{+,-\}$, $a\in\alpha_+$, where $\hat{y}$ is the function defined in (2.12).

When M is large, the matrix elements of $\widetilde{\mathcal{S}}$ reduce to

$$\widetilde{\mathcal{S}}_{ab}\underset{M\to\infty}{\sim}\begin{cases}2i\pi M\sinh\left(\lambda_a-\xi_-+\eta/2\right)\Psi\left(\lambda_a,\xi_b\right) & \text{if } \lambda_a=\check{\lambda}\ ,\\ \rho^{-1}\left(\lambda_a\right)\ \Psi\left(\lambda_a,\xi_b\right) & \text{if } \lambda_a\neq\check{\lambda}\ ,\end{cases} \tag{3.17}$$

the corrections being of order $\mathrm{O}\left(1/M\right)$, and

$$\Psi\left(\lambda,\xi\right)=\frac{\rho\left(\lambda-\xi\right)-\rho\left(\lambda-\eta+\xi\right)}{2\sinh\left(2\xi-\eta\right)}\ . \tag{3.18}$$

The determinant structure of the scalar product as well as peculiarities of the coefficients C_{α_m} enable us to write:

$$\langle \mathcal{O}_m \rangle = \frac{1}{M^m} \sum_{\substack{\{\nu\}_I \subset \{\lambda\}_1^N \cup \{\xi\}_1^m \\ |I|=m}} H_m\left(\{\nu\}_I, \{\xi\}_1^m\right)\left(1 + \mathrm{O}\left(1/M\right)\right), \qquad (3.19)$$

in which the coefficient $H_m\left(\{\nu\}_I, \{\xi\}_1^m\right)$ can be computed generically. Taking the thermodynamic limit $M \to +\infty$ we recast the sums over replaced rapidities λ into integrals:

$$\frac{1}{M} \sum_{i=1}^{N} \sum_{\sigma_i = \pm} \sigma_i\, f\left(\lambda_i^\sigma\right) \underset{M \to +\infty}{\longrightarrow} \int_{\mathscr{C}} \mathrm{d}\lambda\, \rho\left(\lambda\right) f\left(\lambda\right), \qquad \forall f \in \mathcal{C}^0\left(\mathscr{C}\right). \qquad (3.20)$$

In the boundary model the contour of integration $\mathscr{C}$ depends on the anisotropy parameter Δ and on the boundary field h_-.

4 Action of local operators on boundary states

Using Corollary 3.1, Lemma 1 and the boundary-bulk decomposition of Proposition 3.2, it is easy to compute the action of a product of elementary matrices of the form (3.5) on an arbitrary boundary state. This computation was explicitely performed in [4], and enabled us there to obtain some expressions for the elementary building blocks of correlation functions.

The aim of the present article is to obtain such expressions for physical correlation functions, and in particular for one-point functions. Therefore, if we want to use the method recalled in Section 3, the main problem is to obtain some resummed formulas directly for the action of the local spin operators we consider. This is the purpose of the present section.

In the first part of this section, we derive the action of the operator

$$\mathcal{Q}_m\left(\kappa\right) \equiv \prod_{i=1}^{m}\left(E_i^{11} + \kappa E_i^{22}\right) = \prod_{i=1}^{m}\left(A + \kappa D\right)\left(\xi_i\right) \prod_{i=1}^{m}\left(A + D\right)^{-1}\left(\xi_i\right) \qquad (4.1)$$

on arbitrary boundary states. $\langle \mathcal{Q}_m\left(\kappa\right)\rangle$ can be interpreted in the boundary model as the generating function of the magnetization at a distance m from the boundary:

$$\langle \frac{1-\sigma_m^z}{2} \rangle = D_m \partial_\kappa \langle \mathcal{Q}_m\left(\kappa\right)\rangle|_{\kappa=1}, \qquad (4.2)$$

where D_m is the lattice derivative : $D_m\, u_m \equiv u_{m+1} - u_m$.

Then, in the second part of this section, we give the formulas for the action of the local spin operators $E_m^{22} = \frac{1-\sigma_m^z}{2}$, $E_m^{12} = \sigma_m^+$ and $E_m^{21} = \sigma_m^-$ on arbitrary boundary states. Note that the action of E_m^{11} follows from the fact that $E_m^{11} = 1 - E_m^{22}$.

4.1 Action of $\mathcal{Q}_m(\kappa)$

We start by computing the action of $\mathcal{Q}_m(\kappa)$ on an arbitrary bulk state, and then infer from this formula its action on arbitrary boundary states.

Proposition 4.1 *The action of $\mathcal{Q}_m(\kappa)$ on an arbitrary bulk state $|\{\lambda\}_1^N\rangle$ can be expressed as*

$$\mathcal{Q}_m(\kappa)\,|\,\{\lambda\}_1^N\,\rangle = \sum_{n=0}^{m}\sum_{\mathcal{P}_\lambda\,;\,\mathcal{P}_\xi} R_n^\kappa\left(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi\right)\ |\,\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\,\rangle\,. \tag{4.3}$$

In the above formula, we sum over all possible partitions $\mathcal{P}_\lambda$ and $\mathcal{P}_\xi$ of the sets $\{\lambda\}_1^N$ and $\{\xi\}_1^m$ into subsets $\{\lambda\}_{\alpha_+}\cup\{\lambda\}_{\alpha_-}$ and $\{\xi\}_{\gamma_+}\cup\{\xi\}_{\gamma_-}$ respectively, satisfying the constraint on the cardinality $|\,\alpha_+\,|=|\,\gamma_+\,|=n$:

$$\mathcal{P}_\lambda:\ \{\lambda\}_1^N=\{\lambda\}_{\alpha_+}\cup\{\lambda\}_{\alpha_-}\,,\qquad |\,\alpha_+\,|=n\,, \tag{4.4}$$

$$\mathcal{P}_\xi:\ \{\xi\}_1^m=\{\xi\}_{\gamma_+}\cup\{\xi\}_{\gamma_-}\,,\qquad |\,\gamma_+\,|=n\,. \tag{4.5}$$

The coefficient $R_n^\kappa\left(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi\right)$ splits into two parts,

$$R_n^\kappa\left(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi\right)=R\left(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi\right)\ S_n^\kappa\big(\,\{\xi\}_{\gamma_+}\,,\{\lambda\}_{\alpha_+}\,\big)\,, \tag{4.6}$$

the first one having a product structure,

$$R\left(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi\right)=\frac{\prod\limits_{a\in\alpha_+}\left\{a\left(\lambda_a\right)\prod\limits_{b\in\alpha_-}f\left(\lambda_b,\lambda_a\right)\right\}}{\prod\limits_{a\in\gamma_+}\left\{a\left(\xi_a\right)\prod\limits_{b\in\alpha_-\cup\alpha_+}f\left(\lambda_b,\xi_a\right)\right\}}\ \prod_{a\in\gamma_-}\frac{\prod\limits_{b\in\gamma_+}f\left(\xi_b,\xi_a\right)}{\prod\limits_{b\in\alpha_+}f\left(\lambda_b,\xi_a\right)}, \tag{4.7}$$

and the second one, which depends here only on the subsets $\{\lambda\}_{\alpha_+}$ and $\{\xi\}_{\gamma_+}$, being given as a ratio of two determinants,

$$S_n^\kappa\left(\{\nu\}_1^n,\{\mu\}_1^n\right)=\det{}_n\left[M_\kappa\big(\,\{\mu\}_1^n\,,\{\nu\}_1^n\,\big)\right]\ \det{}_n^{-1}\left(\frac{1}{\sinh\left(\nu_k-\mu_j+\eta\right)}\right)\,. \tag{4.8}$$

The entries of the matrix M_κ read

$$\left[M_\kappa\big(\,\{\mu\}_1^n\,,\{\nu\}_1^n\,\big)\right]_{jk}=t\left(\nu_k,\mu_j\right)-\kappa\,t\left(\mu_j,\nu_k\right)\prod_{\substack{a=1\\a\neq j}}^{n}\frac{f\left(\mu_a,\mu_j\right)}{f\left(\mu_j,\mu_a\right)}\prod_{a=1}^{n}\frac{f\left(\mu_j,\nu_a\right)}{f\left(\nu_a,\mu_j\right)}\,, \tag{4.9}$$

and the functions f and t stand for

$$t\left(\lambda,\mu\right)=\frac{\sinh\eta}{\sinh\left(\lambda-\mu\right)\sinh\left(\lambda-\mu+\eta\right)},\qquad f\left(\lambda,\mu\right)=\frac{\sinh\left(\lambda-\mu+\eta\right)}{\sinh\left(\lambda-\mu\right)}. \tag{4.10}$$

The above theorem appears as a non-trivial generalization of the action of $\mathcal{Q}_m(\kappa)$ on bulk Bethe eigenvectors [49]. Indeed, when $|\{\lambda\}\rangle$ is not an eigenstate of the bulk transfer matrix, then $\prod_{i=1}^m (A+D)(\xi_i)$ does not act by multiplication any more. Of course our result reproduces the previous case when we send the parameters λ to a solution of the bulk Bethe equations.

Proof — The proof goes by induction on m.

Property (4.3) is obvious for $m=1$. Assume that it holds for some m. To prove its validity for $m+1$ we have to compute

$$\mathcal{Q}_{m+1}(\kappa)\,|\,\{\lambda\}_1^N\,\rangle = \frac{(A+\kappa D)(\xi_{m+1})\,\mathcal{Q}_m(\kappa)\,(A+D)(\xi_{m+1}-\eta)}{a(\xi_{m+1})d(\xi_{m+1}-\eta)}|\,\{\lambda\}_1^N\,\rangle. \qquad (4.11)$$

Let us first reproduce the coefficient $R_n^\kappa(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi)$ in the case when the partition $\mathcal{P}_\xi$ is such that $\xi_{m+1}\notin\{\xi\}_{\gamma_+}$. The corresponding state $|\,\{\lambda\}_{\alpha_-}\cup\{\xi\}_{\gamma_+}\,\rangle$ can only be obtained by the direct action of $(A+\kappa D)(\xi_{m+1})$. In order to reproduce the claimed form of the coefficient R_n^κ it is enough to prove that $(A+\kappa D)(\xi_{m+1}-\eta)$ acts directly. Suppose that this is not the case. Then $\mathcal{Q}_m(\kappa)$ acts on a state containing $\xi_{m+1}-\eta$. In virtue of Lemma 1, the action of $\mathcal{Q}_m(\kappa)$ on these states cannot replace $\xi_{m+1}-\eta$. Thus $(A+D)(\xi_{m+1})$ exchanges $\xi_{m+1}-\eta$ with ξ_{m+1}, which leads to a contradiction.

We still have to reproduce the coefficient $R_n^\kappa(\mathcal{P}_\lambda\,,\,\mathcal{P}_\xi)$ corresponding to states $|\,\{\lambda\}_{\alpha_-}\cup\{\xi\}_{\gamma_+}\,\rangle$ such that $\xi_{m+1}\in\{\xi\}_{\gamma_+}$. Theorem 3.1 yields the decomposition:

$$\mathcal{Q}_{m+1}(\kappa) = \frac{A(\xi_{m+1})\,\mathcal{Q}_m(\kappa)}{a(\xi_{m+1})d(\xi_{m+1}-\eta)}\underbrace{D(\xi_{m+1}-\eta)}_{(1)} + \frac{\kappa D(\xi_{m+1})\,\mathcal{Q}_m(\kappa)}{a(\xi_{m+1})d(\xi_{m+1}-\eta)}\underbrace{A(\xi_{m+1}-\eta)}_{(2)},$$

whereas Lemma 1 ensures that

- (1) only acts directly; indeed $A(\xi_{m+1})$ cannot replace $\xi_{m+1}-\eta$ by ξ_{m+1};
- (2) acts indirectly and thus $D(\xi_{m+1})$ only acts by substitution.

The formula for R_n^κ (4.6) follows after computing the resulting actions and rearranging the sums thanks to the re-summation formula provided by the contour integral:

$$0 = \oint\limits_{\mathbb{R}\cup\mathbb{R}+i\pi} \frac{\mathrm{d}z}{\sinh(z-\xi_{n+1})}\prod_{a=1}^{r}\frac{f(z,x_a)}{f(\xi_{n+1},x_a)}\;S_{n+1}^\kappa\big(\{\xi\}_1^n\cup\{z\}\,,\{\lambda\}_1^{n+1}\big)\quad. \qquad (4.12)$$

Note that the parameters x_a appearing in the contour integral (4.12) are generic. □

Using the boundary-bulk decomposition of Proposition 3.2, one can now deduce from Proposition 4.1 the action of $\mathcal{Q}_m(\kappa)$ on arbitrary boundary states.

Corollary 4.1 *The action of $\mathcal{Q}_m(\kappa)$ on an arbitrary boundary state $|\,\{\lambda\}_1^N\,\rangle_b$ reads:*

$$\mathcal{Q}_m(\kappa)\,|\,\{\lambda\}_1^N\,\rangle_b = \sum_{n=0}^{m}\sum_{\mathcal{P}_\lambda\,;\,\mathcal{P}_\xi}\mathcal{R}_n^\kappa(\mathcal{P}_\lambda,\mathcal{P}_\xi)\;\,|\,\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\,\rangle_b. \qquad (4.13)$$

The sum over partitions is defined as in Theorem 4.1, and the coefficient $\mathcal{R}_n^\kappa$ can be expressed as

$$\mathcal{R}_n^\kappa\,(\mathcal{P}_\lambda,\mathcal{P}_\xi)=\sum_{\substack{\sigma_i=\pm\\ i\in\alpha_+}}\mathcal{R}_\sigma\,(\mathcal{P}_\lambda,\mathcal{P}_\xi)\;\;S_n^\kappa(\{\xi\}_{\gamma_+},\{\lambda^\sigma\}_{\alpha_+})\,,\tag{4.14}$$

where $S_n^\kappa\,(\{\nu\}_1^n\,,\{\mu\}_1^n)$ is the bulk function defined in (4.8)*, while $\mathcal{R}_\sigma\,(\mathcal{P}_\lambda,\mathcal{P}_\xi)$ is the boundary dressing of* (4.7)*:*

$$\mathcal{R}_\sigma(\mathcal{P}_\lambda,\mathcal{P}_\xi)=\frac{\prod\limits_{a\in\alpha_+}\Big\{a(\lambda_a^\sigma)\prod\limits_{b\in\alpha_-}\big[f(\lambda_b,\lambda_a^\sigma)\,f(-\lambda_b,\lambda_a^\sigma)\big]\Big\}}{\prod\limits_{a\in\gamma_+}\Big\{a(\xi_a)\prod\limits_{b\in\alpha_+}f(\lambda_b^\sigma,\xi_a)\prod\limits_{b\in\alpha_-}\big[f(\lambda_b,\xi_a)f(-\lambda_b,\xi_a)\big]\Big\}}\\ \times\prod_{a\in\gamma_-}\frac{\prod\limits_{b\in\gamma_+}f(\xi_b,\xi_a)}{\prod\limits_{b\in\alpha_+}f(\lambda_b^\sigma,\xi_a)}\;\frac{H^{\mathcal{B}}_{\{\sigma\}_{\alpha_+}}(\{\lambda\}_{\alpha_+})}{H^{\mathcal{B}}(\{\xi\}_{\gamma_+})}.\tag{4.15}$$

Here $H^{\mathcal{B}}_{\{\sigma\}_{\alpha_+}}(\{\lambda\}_{\alpha_+})$ and $H^{\mathcal{B}}(\{\xi\}_{\gamma_+})$ stand for the boundary-bulk coefficients (3.8) *associated respectively to $\{\lambda\}_{\alpha_+}$, $\{\sigma\}_{\alpha_+}$, and to $\{\xi\}_{\gamma_+}$, $\{\sigma\}_{\gamma_+}=\{1,\ldots,1\}$.*

Proof — The proof is a straightforward consequence of the boundary-bulk decomposition (3.7) applied to Proposition 4.1. More precisely, expressing the boundary state $|\{\lambda\}_1^N\rangle_b$ in terms of the bulk states $|\{\lambda^\sigma\}_1^N\rangle$, and using (4.3), we get

$$\mathcal{Q}_m\,(\kappa)\,|\,\{\lambda\}_1^N\rangle_b=\sum_{n=0}^m\sum_{\mathcal{P}_\lambda\,;\,\mathcal{P}_\xi}\sum_{\substack{\sigma_i=\pm\\ 1\le i\le N}}H^{\mathcal{B}}_{\{\sigma\}}(\{\lambda\}_1^N)\;R_n^\kappa\,(\mathcal{P}_{\lambda^\sigma}\,,\,\mathcal{P}_\xi)\;\;|\,\{\xi\}_{\gamma_+}\cup\{\lambda^\sigma\}_{\alpha_-}\rangle.$$

We now use the fact that

$$H^{\mathcal{B}}_{\{\sigma\}}(\{\lambda\}_1^N)=\prod_{b\in\alpha_-}\frac{\prod\limits_{a\in\alpha_+}f(-\lambda_b^\sigma,\lambda_a^\sigma)}{\prod\limits_{a\in\gamma_+}f(-\lambda_b^\sigma,\xi_a)}\;\frac{H^{\mathcal{B}}_{\{\sigma\}_{\alpha_+}}(\{\lambda\}_{\alpha_+})}{H^{\mathcal{B}}(\{\xi\}_{\gamma_+})}\;H^{\mathcal{B}}_{1,\{\sigma\}_{\alpha_-}}(\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}),$$

where $H^{\mathcal{B}}_{1,\{\sigma\}_{\alpha_-}}(\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-})$ is the boundary-bulk coefficient of $|\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\rangle_b$ in terms of $|\{\xi\}_{\gamma_+}\cup\{\lambda^\sigma\}_{\alpha_-}\rangle$. Note that the first factor of this product combines with the products over $b\in\alpha_-$ in the expression (4.7) of $R(\mathcal{P}_{\lambda^\sigma},\mathcal{P}_\xi)$, and that the resulting factor,

$$\prod_{b\in\alpha_-}\frac{\prod\limits_{a\in\alpha_+}\big[f(\lambda_b^\sigma,\lambda_a^\sigma)\,f(-\lambda_b^\sigma,\lambda_a^\sigma)\big]}{\prod\limits_{a\in\gamma_+}\big[f(\lambda_b^\sigma,\xi_a)\,f(-\lambda_b^\sigma,\xi_a)\big]},$$

is actually independant of the value of σ_i for $i\in\alpha_-$. It enables us to reconstruct the boundary state $|\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\rangle_b$, with a coefficient which reduces to (4.14). □

4.2 Action of local spin operators

We list here the action of the local spin operators σ_m^-, σ_m^+ and E_m^{22} on bulk and boundary states. We omit the proofs since, although a little more technical, they parallel the one concerning the action of $\mathcal{Q}_m(\kappa)$.

Proposition 4.2 *The action of* σ_m^-, E_m^{22} *and* σ_m^+ *on an arbitrary bulk state* $|\{\lambda\}_1^N\rangle$ *can be expressed as*

$$\sigma_m^- \, |\{\lambda\}_1^N\rangle = \sum_{n=0}^{m-1} \sum_{\mathcal{P}_\lambda^-, \mathcal{P}_\xi} R_n^-(\mathcal{P}_\lambda^-, \mathcal{P}_\xi) \; |\{\xi\}_{\gamma_+} \cup \{\lambda\}_{\alpha_-}\rangle,$$

$$E_m^{22} \, |\{\lambda\}_1^N\rangle = \sum_{n=0}^{m-1} \sum_{c_1=1}^{N} \sum_{\mathcal{P}_\lambda^{22}, \mathcal{P}_\xi} R_n^{22}(\mathcal{P}_\lambda^{22}, \mathcal{P}_\xi) \; |\{\xi\}_{\gamma_+} \cup \{\lambda\}_{\alpha_-}\rangle,$$

$$\sigma_m^+ \, |\{\lambda\}_1^N\rangle = \lim_{\lambda_{N+1} \to \xi_m} \sum_{n=0}^{m-1} \sum_{c_1=1}^{N} \sum_{\substack{c_2=1 \\ c_2 \neq c_1}}^{N+1} \sum_{\mathcal{P}_\lambda^+, \mathcal{P}_\xi} R_n^+(\mathcal{P}_\lambda^+, \mathcal{P}_\xi) \; |\{\xi\}_{\gamma_+} \cup \{\lambda\}_1^N \setminus \{\lambda\}_{\widetilde{\alpha}_+}\rangle,$$

in which the sums run over the following partitions

$$\mathcal{P}_\xi \;:\; \{\xi\}_1^m = \{\xi\}_{\gamma_+} \cup \{\xi\}_{\gamma_-}, \qquad \textit{with} \quad |\gamma_+| = n+1, \tag{4.16}$$

$$\mathcal{P}_\lambda^- \;:\; \{\lambda\}_1^N = \{\lambda\}_{\alpha_+} \cup \{\lambda\}_{\alpha_-}, \qquad \textit{with} \quad |\alpha_+| = n, \tag{4.17}$$

$$\mathcal{P}_\lambda^{22} \;:\; \{\lambda_k\}_{\substack{1 \le k \le N \\ k \neq c_1}} = \{\lambda\}_{\alpha_+} \cup \{\lambda\}_{\alpha_-}, \qquad \textit{with} \quad |\alpha_+| = n, \tag{4.18}$$

$$\mathcal{P}_\lambda^+ \;:\; \{\lambda_k\}_{\substack{1 \le k \le N \\ k \neq c_1, c_2}} = \{\lambda\}_{\alpha_+} \cup \{\lambda\}_{\alpha_-}, \qquad \textit{with} \quad |\alpha_+| = n. \tag{4.19}$$

We also define the following partitions, associated respectively to (4.18) *and to* (4.19),

$$\widetilde{\mathcal{P}}_\lambda^{22} \;:\; \{\lambda\}_1^N = \{\lambda\}_{\widetilde{\alpha}_+} \cup \{\lambda\}_{\alpha_-}, \qquad \textit{with} \quad \widetilde{\alpha}_+ = \alpha_+ \cup \{c_1\}, \tag{4.20}$$

$$\widetilde{\mathcal{P}}_\lambda^+ \;:\; \{\lambda\}_1^{N+1} = \{\lambda\}_{\widetilde{\alpha}_+} \cup \{\lambda\}_{\widetilde{\alpha}_-}, \qquad \textit{with} \quad \widetilde{\alpha}_+ = \alpha_+ \cup \{c_1, c_2\}. \tag{4.21}$$

The coefficients $R_n^-(\mathcal{P}_\lambda^-, \mathcal{P}_\xi)$, $R_n^{22}(\mathcal{P}_\lambda^{22}, \mathcal{P}_\xi)$ *and* $R_n^+(\mathcal{P}_\lambda^+, \mathcal{P}_\xi)$ *are given as*

$$R_n^-(\mathcal{P}_\lambda^-, \mathcal{P}_\xi) = R(\mathcal{P}_\lambda^-, \mathcal{P}_\xi) \; \lim_{\xi \to \xi_m} \frac{\prod_{a \in \gamma_+} \sinh(\xi_a - \xi)}{\prod_{a \in \alpha_+} \sinh(\lambda_a - \xi)} \; \widehat{S}_n(\{\lambda\}_{\alpha_+}, \{\xi\}_{\gamma_+}; \xi, \emptyset), \tag{4.22}$$

$$\begin{aligned} R_n^{22}(\mathcal{P}_\lambda^{22}, \mathcal{P}_\xi) = R(\widetilde{\mathcal{P}}_\lambda^{22}, \mathcal{P}_\xi) \; \sinh\eta \prod_{a \in \alpha_+} f(\lambda_a, \lambda_{c_1}) & \\ \times \lim_{\xi \to \xi_m} \frac{\prod_{a \in \gamma_+} \sinh(\xi_a - \xi)}{\prod_{a \in \widetilde{\alpha}_+} \sinh(\lambda_a - \xi)} \; \widehat{S}_n(\{\lambda\}_{\alpha_+}, \{\xi\}_{\gamma_+}; \xi, \{\lambda_{c_1}\}), & \end{aligned} \tag{4.23}$$

$$R_n^{+}\big(\mathcal{P}_\lambda^{+}\,,\,\mathcal{P}_\xi\big) = R\big(\widetilde{\mathcal{P}}_\lambda^{+}\,,\,\mathcal{P}_\xi\big)\; f(\lambda_{c_2},\lambda_{c_1})\,\prod_{i=1}^{2}\Big\{\sinh\eta\prod_{a\in\alpha_+} f(\lambda_a,\lambda_{c_i})\Big\}$$
$$\times\frac{\prod\limits_{a\in\gamma_+}\sinh(\lambda_{N+1}-\xi_a+\eta)}{\prod\limits_{a\in\tilde{\alpha}_+}\sinh(\lambda_{N+1}-\lambda_a+\eta)}\;\widehat{S}_n\big(\,\{\xi\}_{\gamma_+}\,,\{\lambda\}_{\alpha_+}\,;\lambda_{N+1},\{\lambda_{c_1},\lambda_{c_2}\}\big). \quad (4.24)$$

Here $R(\mathcal{P}_\lambda,\mathcal{P}_\xi)$ *is given by* (4.7)*, and the structure of the factor* $\widehat{S}_n(\{\xi\}_1^{n+1},\{\lambda\}_1^n;\xi,\{\mu\}_1^p)$ *is similar to* (4.8)*:*

$$\widehat{S}_n(\{\xi\}_1^{n+1},\{\lambda\}_1^n;\xi,\{\mu\}_1^p) = \frac{\prod\limits_{a=1}^{n}\prod\limits_{b=1}^{n+1}\sinh(\xi_b-\lambda_a+\eta)}{\prod\limits_{a>b}\sinh(\xi_a-\xi_b)\prod\limits_{a>b}\sinh(\lambda_b-\lambda_a)}$$
$$\times\det{}_{n+1}\Big[\widehat{M}(\{\lambda\}_1^n,\{\xi\}_1^{n+1};\xi,\{\mu\}_1^p)\Big], \quad (4.25)$$

where the matrix elements of $\widehat{M}$ *are obtained as*

$$\big[\widehat{M}\big(\{\lambda\}_1^n,\{\xi\}_1^{n+1};\lambda_{n+1},\{\mu\}_1^p\big)\big]_{jk} = \big[M_{\kappa(\lambda_j,\{\mu\})}\big(\,\{\lambda\}_1^{n+1}\,,\{\xi\}_1^{n+1}\,\big)\big]_{jk}, \quad (4.26)$$

with

$$\kappa(\lambda_j,\{\mu\}) = (1-\delta_{j,n+1})\prod_{i=1}^{n'}\frac{f\,(\mu_i,\lambda_j)}{f\,(\lambda_j,\mu_i)},$$

in which δ_{ij} *denotes the Kronecker symbol and* M_κ *is defined as in* (4.9)*.*

Using again the boundary-bulk decomposition, we are now in position to list the action of local spin operators on boundary states.

Corollary 4.2 *With the same notations as in Proposition 4.2, the action of* σ_m^-*,* E_m^{22} *and* σ_m^+ *on an arbitrary boundary state* $|\{\lambda\}_1^N\,\rangle_b$ *takes the form*

$$\sigma_m^-\,|\{\lambda\}_1^N\,\rangle_b = \sum_{n=0}^{m-1}\sum_{\mathcal{P}_\lambda^-,\mathcal{P}_\xi}\mathcal{R}_n^-(\mathcal{P}_\lambda^-,\mathcal{P}_\xi)\;|\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\,\rangle_b,$$

$$E_m^{22}\,|\{\lambda\}_1^N\,\rangle_b = \sum_{n=0}^{m-1}\sum_{c_1=1}^{N}\sum_{\mathcal{P}_\lambda^{22},\mathcal{P}_\xi}\mathcal{R}_n^{22}(\mathcal{P}_\lambda^{22},\mathcal{P}_\xi)\;|\{\xi\}_{\gamma_+}\cup\{\lambda\}_{\alpha_-}\,\rangle_b,$$

$$\sigma_m^+\,|\{\lambda\}_1^N\,\rangle_b = \lim_{\lambda_{N+1}\to\xi_m}\sum_{n=0}^{m-1}\sum_{c_1=1}^{N}\sum_{\substack{c_2=1\\ c_2\neq c_1}}^{N+1}\sum_{\mathcal{P}_\lambda^+,\mathcal{P}_\xi}\mathcal{R}_n^+(\mathcal{P}_\lambda^+,\mathcal{P}_\xi)\;|\,\{\xi\}_{\gamma_+}\cup\{\lambda\}_1^N\setminus\{\lambda\}_{\widetilde{\alpha}_+}\,\rangle_b.$$

The boundary coefficients $\mathcal{R}^-$*,* $\mathcal{R}^{22}$ *and* $\mathcal{R}^+$ *have a structure similar to their corresponding bulk counterparts:*

$$\mathcal{R}_n^-(\mathcal{P}_\lambda^-,\mathcal{P}_\xi) = \sum_{\substack{\sigma_i=\pm\\ i\in\alpha_+}}\mathcal{R}_\sigma(\mathcal{P}_\lambda^-,\mathcal{P}_\xi)\;\lim_{\xi\to\xi_m}\frac{\prod\limits_{a\in\gamma_+}\sinh(\xi_a-\xi)}{\prod\limits_{a\in\alpha_+}\sinh(\lambda_a-\xi)}\;\widehat{S}_n(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\xi,\emptyset),$$

$$\mathcal{R}_n^{22}(\mathcal{P}_\lambda^{22},\mathcal{P}_\xi)=\sum_{\substack{\sigma_i=\pm\\ i\in\widetilde{\alpha}_+}}\mathcal{R}_\sigma(\widetilde{\mathcal{P}}_\lambda^{22},\mathcal{P}_\xi)\ \sinh\eta\prod_{a\in\alpha_+}f(\lambda_a,\lambda_{c_1})$$
$$\times\lim_{\xi\to\xi_m}\frac{\prod\limits_{a\in\gamma_+}\sinh(\xi_a-\xi)}{\prod\limits_{a\in\widetilde{\alpha}_+}\sinh(\lambda_a-\xi)}\ \widehat{S}_n(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\xi,\{\lambda_{c_1}\}),$$

$$\mathcal{R}_n^{+}(\mathcal{P}_\lambda^{+},\mathcal{P}_\xi)=\sum_{\substack{\sigma_i=\pm\\ i\in\widetilde{\alpha}_+}}\mathcal{R}_\sigma(\widetilde{\mathcal{P}}_\lambda^{+},\mathcal{P}_\xi)\ f\left(\lambda_{c_2}^\sigma,\lambda_{c_1}^\sigma\right)\prod_{i=1}^{2}\left\{\sinh\eta\prod_{a\in\alpha_+}f\left(\lambda_a^\sigma,\lambda_{c_i}^\sigma\right)\right\}$$
$$\times\frac{\prod\limits_{a\in\gamma_+}\left[f(-\lambda_{N+1},\xi_a)\sinh(\lambda_{N+1}-\xi_a+\eta)\right]}{\prod\limits_{a\in\widetilde{\alpha}_+}\left[f(-\lambda_{N+1},\lambda_a^\sigma)\sinh(\lambda_{N+1}-\lambda_a^\sigma+\eta)\right]}\ \widehat{S}_n\left(\{\xi\}_{\gamma_+},\{\lambda^\sigma\}_{\alpha_+};\lambda_{N+1},\{\lambda_{c_i}^\sigma\}\right),$$

where $\mathcal{R}_\sigma$ is defined as in (4.15) *and $\widehat{S}_n$ is the bulk quantity* (4.25)*.*

5 Correlation functions in the half-infinite chain

We apply the results of the previous section to derive the expectation values of the generating function $\langle\mathcal{Q}_m(\kappa)\rangle$ of $\langle\sigma_m^z\rangle$, and of $\langle\sigma_1^+\sigma_{m+1}^-\rangle$ in the ground state of the half-infinite chain. These are the boundary analogues of the results published in [49].

5.1 The generating function $\langle\mathcal{Q}_m(\kappa)\rangle$

Proposition 5.1 *The generating function $\langle\mathcal{Q}_m(\kappa)\rangle$ is obtained, in the thermodynamic limit $M\to+\infty$, as the homogeneous limit of the quantity*

$$\langle\mathcal{Q}_m(\kappa)\rangle=\sum_{n=0}^{m}\frac{1}{(n!)^2}\oint\limits_{\Gamma_+(\{\xi\}_1^m)}\frac{\mathrm{d}^n z}{(2i\pi)^n}\int\limits_{\mathcal{C}_D}\mathrm{d}^n\lambda\ \prod_{a=1}^{m}\prod_{b=1}^{n}\frac{f(z_b,\xi_a)}{f(\lambda_b,\xi_a)}\ \mathcal{W}_-(\{\lambda\}_1^n,\{z\}_1^n)$$
$$\times\det\nolimits_n\left[M_\kappa(\{\lambda\},\{z\})\right]\ \det\nolimits_n\left[\Psi(\lambda_j,z_k)\right],\qquad(5.1)$$

in which M_κ is given by (4.9)*, and $\mathcal{W}_-$ is the boundary dressing,*

$$\mathcal{W}_-\left(\{\lambda\}_1^{n_1},\{z\}_1^{n_2}\right)=\frac{\prod\limits_{j=1}^{n_2}\sinh(z_j+\xi_--\eta/2)}{\prod\limits_{j=1}^{n_1}\sinh(\lambda_j+\xi_--\eta/2)}$$
$$\times\frac{\prod\limits_{a=1}^{n_1}\prod\limits_{b=1}^{n_2}\sinh(z_b+\lambda_a-\eta)}{\prod\limits_{a<b}^{n_2}\sinh(\overline{z}_{ab}-\eta)\prod\limits_{a<b}^{n_1}\sinh\left(\overline{\lambda}_{ab}-\eta\right)}\ W\left(\{\lambda\}_1^{n_1},\{z\}_1^{n_2}\right),\quad(5.2)$$

of the bulk quantity

$$W\left(\{\lambda\}_1^{n_1},\{z\}_1^{n_2}\right)=\frac{\prod\limits_{a=1}^{n_1}\prod\limits_{b=1}^{n_2}\left[\sinh\left(z_b-\lambda_a-\eta\right)\sinh\left(z_b-\lambda_a+\eta\right)\right]}{\prod\limits_{a,b=1}^{n_1}\sinh\left(\lambda_{ab}-\eta\right)\prod\limits_{a,b=1}^{n_2}\sinh\left(z_{ab}+\eta\right)}. \tag{5.3}$$

The contour of integration $\mathcal{C}_D$ depends on the boundary magnetic field h_-:

$$\mathcal{C}_D=\begin{cases}]-\Lambda\,;\Lambda\,[\,\cup\Gamma_+\left(\check{\lambda}\right) & \textit{if}\quad 0<\tilde{\xi}_-<\zeta/2,\\ ]-\Lambda\,;\Lambda\,[& \textit{otherwise},\end{cases} \tag{5.4}$$

where $\Gamma_\pm(z)$ stands for a small loop of index ±1 with respect to z. We recall that $\Lambda=+\infty$ for $-1<\Delta<1$ and $\Lambda=-i\pi/2$ for $\Delta>1$.

Proof — Corollary 4.1 yields the action of $\mathcal{Q}_m(\kappa)$ on a boundary state. It is convenient to note that the coefficient $\mathcal{R}_\sigma\big(\mathcal{P}_\lambda,\mathcal{P}_\xi\big)$ (4.15) can be rewritten as

$$\mathcal{R}_\sigma\big(\mathcal{P}_\lambda,\mathcal{P}_\xi\big)=\left(\prod_{a\in\alpha_+}\sigma_a\right)(\sinh\eta)^{|\gamma_+|}\prod_{b\in\gamma_+\cup\gamma_-}\frac{\prod\limits_{\substack{a\in\gamma_+\\ a\neq b}}f(\xi_a,\xi_b)}{\prod\limits_{a\in\alpha_+}f(\lambda_a^\sigma,\xi_b)}\quad\mathcal{W}_-(\{\lambda^\sigma\}_{\alpha_+},\{\xi\}_{\gamma_+})$$
$$\times\frac{\prod\limits_{a>b}\sinh(\xi_a-\xi_b)\prod\limits_{a>b}\sinh(\lambda_b-\lambda_a)}{\prod\limits_{a\in\alpha_+}\prod\limits_{b\in\gamma_+}\sinh(\xi_b-\lambda_a+\eta)}\;\mathcal{S}_\sigma(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\{\lambda\}_{\alpha_-})^{-1}, \tag{5.5}$$

in which $\mathcal{S}_\sigma(\{\lambda\}_{\alpha_+},\{\xi\}_{\gamma_+};\{\lambda\}_{\alpha_-})$ is the function defined in (3.16). Then, using the reduced scalar product formula (3.15) and absorbing the sums over partitions $\mathcal{P}_\xi$ into auxiliary z integrals[5], we obtain the former representation.

Note that the contour contains $\Gamma_+\left(\check{\lambda}\right)$ for large positive boundary field since we have to absorb the contribution coming from the replacement of the complex root $\check{\lambda}$ as explained in [4]. □

[5]We refer the reader to [49] for technical details.

5.2 The ground state expectation value $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$

Using the same method as for the generating function $\langle \mathcal{Q}_m(\kappa) \rangle$, we can also compute the ground state expectation value $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$. It gives

$$\begin{aligned}\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{\sinh(\xi_1 + \xi_- - \eta/2)}{n!(n+1)!} & \oint\limits_{\Gamma_+(\{\xi\}_1^{m+1})} \prod_{k=1}^{n+1} \frac{\mathrm{d} z_k}{2i\pi} \int\limits_{\mathcal{C}_D} \prod_{k=1}^{n+1} \mathrm{d}\lambda_k \int\limits_{\mathcal{C}_A} \mathrm{d}\lambda_{n+2} \\ & \times \prod_{a=2}^{m+1} \frac{\prod_{b=1}^{n+1} f(z_b, \xi_a)}{\prod_{b=1}^{n} f(\lambda_b, \xi_a)} \prod_{b=1}^{n} \frac{\sinh(\lambda_b - \xi_1)}{\sinh(\lambda_b - \xi_{m+1})} \prod_{b=1}^{n+1} \frac{\sinh(z_b - \xi_{m+1})}{\sinh(z_b - \xi_1)} \\ & \times \frac{\prod_{b=1}^{n+1} \sinh(\lambda_b - \lambda_{n+1} + \eta) \prod_{b=1}^{n+2} \sinh(\lambda_{n+2} - \lambda_b + \eta)}{\prod_{b=1}^{n+1} \big[\sinh(z_b - \lambda_{n+1} + \eta) \sinh(\lambda_{n+2} - z_b + \eta) \big]} \mathcal{W}_-\big(\{\lambda\}_1^{n+2}, \{z\}_1^{n+1} \big) \\ & \times \frac{\prod_{b=1}^{n+2} \sinh(\xi_1 + \lambda_b - \eta)}{\prod_{b=1}^{n+1} \sinh(\xi_1 + z_b - \eta)} \det_{n+1} \Big[\widehat{M}\big(\{\lambda\}_1^n, \{z\}_1^{n+1}; \xi_{m+1} \big) \Big] \\ & \times \det_{n+2} [\Psi(\lambda_j, \xi_1), \Psi(\lambda_j, z_1), \ldots, \Psi(\lambda_j, z_{n+1})] . \end{aligned} \tag{5.6}$$

In this expression, $\mathcal{W}_-$ denotes the boundary quantity (5.2), $\widehat{M}\big(\{\lambda\}_1^n, \{z\}_1^{n+1}; \xi_{m+1} \big)$ is a simplified notation for the matrix $\widehat{M}\big(\{\lambda\}_1^n, \{z\}_1^{n+1}; \xi_{m+1}, \emptyset \big)$ defined in (4.26), $\mathcal{C}_D$ is the contour (5.4), and $\mathcal{C}_A$ denotes the following contour (A-type contour):

$$\mathcal{C}_A = \begin{cases} \,]-\Lambda + \eta \,; \Lambda + \eta \, [\, \cup \, \Gamma_-\big(\check{\lambda} \big) , & \text{if} \quad -\zeta/2 < \tilde{\xi}_- < 0, \\ \,]-\Lambda + \eta \,; \Lambda + \eta \, [\,, & \text{otherwise.} \end{cases} \tag{5.7}$$

In the homogeneous limit, this results simplifies into

$$\begin{aligned}\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{\sinh \xi_-}{n!(n+1)!} & \oint\limits_{\Gamma_+(\eta/2)} \prod_{k=1}^{n+1} \frac{\mathrm{d} z_k}{2i\pi} \int\limits_{\mathcal{C}_D} \prod_{k=1}^{n+1} \mathrm{d}\lambda_k \int\limits_{\mathcal{C}_A} \mathrm{d}\lambda_{n+2} \\ & \times \prod_{a=1}^{n+1} \left[\frac{\sinh(z_a + \eta/2)}{\sinh(z_a - \eta/2)} \right]^m \prod_{a=1}^{n} \left[\frac{\sinh(\lambda_a - \eta/2)}{\sinh(\lambda_a + \eta/2)} \right]^m \frac{\prod_{b=1}^{n+2} \sinh(\lambda_b - \eta/2)}{\prod_{b=1}^{n+1} \sinh(z_b - \eta/2)} \\ & \times \frac{\prod_{b=1}^{n+1} \sinh(\lambda_b - \lambda_{n+1} + \eta) \prod_{b=1}^{n+2} \sinh(\lambda_{n+2} - \lambda_b + \eta)}{\prod_{b=1}^{n+1} \big[\sinh(z_b - \lambda_{n+1} + \eta) \sinh(\lambda_{n+2} - z_b + \eta) \big]} \mathcal{W}_-\big(\{\lambda\}_1^{n+2}, \{z\}_1^{n+1} \big) \\ & \times \det_{n+1} \Big[\widehat{M}(\{\lambda\}_1^n, \{z\}_1^{n+1}; \eta/2) \Big] \det_{n+2} [\Psi(\lambda_j, \eta/2), \Psi(\lambda_j, z_1), \ldots, \Psi(\lambda_j, z_{n+1})] , \end{aligned}$$

6 An alternative resummation

6.1 Bulk type resumations

We have obtained in the previous sections a series representation for the generating function. It happens, just as in the bulk case [52], that it is also possible to derive a totally different representation for $\langle \mathcal{Q}_m(\kappa)\rangle$. The latter is based on a re-summation of its expansion with respect to elementary blocks :

$$\langle \mathcal{Q}_m(\kappa)\rangle = \langle \prod_{i=1}^{m} \left(E_i^{11} + \kappa E_i^{22}\right)\rangle = \sum_{s=0}^{m} \kappa^s F_s \ , \tag{6.1}$$

where

$$F_s = \frac{1}{s!\,(m-s)!} \sum_{\pi\in\Sigma_m} \langle E_1^{\epsilon_{\pi(1)}\,\epsilon_{\pi(1)}} \dots E_m^{\epsilon_{\pi(m)}\,\epsilon_{\pi(m)}}\rangle, \quad \epsilon_i = \left\{ \begin{array}{ll} 2, & i = 1...s, \\ 1, & i = s+1...m, \end{array}\right. \tag{6.2}$$

and Σ_m is the group of permutations of m elements. These elementary blocks were computed in [4]. They can be written as multiple integrals in the half-infinite size limit:

$$\begin{aligned}\langle E_1^{\epsilon_1\,\epsilon'_1} \dots E_m^{\epsilon_m\,\epsilon'_m}\rangle &= (-1)^{m-s} \int_{\mathcal{C}_D} \prod_{i=1}^{s} \mathrm{d}\lambda_i \int_{\mathcal{C}_A} \prod_{i=s+1}^{m} \mathrm{d}\lambda_i \, \frac{\det_m\left[\Psi\left(\lambda_i,\xi_j\right)\right]}{\prod\limits_{i<j}^{m}\left[\,\sinh\xi_{ij}\ \sinh\left(\overline{\xi}_{ij}-\eta\right)\,\right]} \\ &\times \frac{\prod\limits_{i,j}^{m}\sinh\left(\lambda_i+\xi_j-\eta\right)}{\prod\limits_{i>j}\sinh\left(\lambda_{ij}-\eta\right)\sinh\left(\overline{\lambda}_{ij}-\eta\right)} \prod_{p=1}^{s}\left[\prod_{j=1}^{i_p-1}\sinh\left(\xi_j-\lambda_p\right)\prod_{j=i_p+1}^{m}\sinh\left(\xi_j-\lambda_p-\eta\right)\right] \\ &\times \prod_{i=1}^{m}\frac{\sinh\left(\xi_i+\xi_- -\eta/2\right)}{\sinh\left(\lambda_i+\xi_- -\eta/2\right)} \prod_{p=s+1}^{m}\left[\prod_{j=1}^{i_p-1}\sinh\left(\xi_j-\lambda_p\right)\prod_{j=i_p+1}^{m}\sinh\left(\xi_j-\lambda_p+\eta\right)\right].\end{aligned} \tag{6.3}$$

The indices i_p are defined by

$$\left\{ \begin{array}{lcl} \{i\ :\ 1\leq i\leq m,\ \epsilon'_i = 2\} & = & \{i_1 < \dots < i_s\}\,, \\ \{i\ :\ 1\leq i\leq m,\ \epsilon_i = 1\} & = & \{i_{s+1} > \dots > i_m\}\,. \end{array}\right. \tag{6.4}$$

For simplicity, we consider from now on the massless regime (although all what follows can be performed in the massive regime as well). In that case, $\eta = -i\zeta$ and the contours of integration $\mathcal{C}_D$ and $\mathcal{C}_A$ depend on the boundary magnetic field h_- as follows:

range of ξ_-	$D-$ contour	$A-$ contour
$\zeta/2 <\mid \tilde{\xi}_- \mid < \pi/2$	$\mathcal{C}_D = \mathbb{R}$	$\mathcal{C}_A = \mathbb{R} - i\zeta$
$\zeta/2 > \tilde{\xi}_- > 0$	$\mathcal{C}_D = \mathbb{R}\bigcup\Gamma_+\left(\check{\lambda}\right)$	$\mathcal{C}_A = \mathbb{R} - i\zeta$
$-\zeta/2 < \tilde{\xi}_- < 0$	$\mathcal{C}_D = \mathbb{R}$	$\mathcal{C}_A = \{\mathbb{R} - i\zeta\}\bigcup\Gamma_-\left(\check{\lambda}\right)$

(6.5)

We recall that $\check{\lambda} = \eta/2 - \xi_-$, and that $\Gamma_\pm(z)$ is a small loop around z of index ± 1.

We now perform a change of variables in the A-type contours: $\lambda'_A = \lambda_A - i\zeta$. Moreover, we shift the inhomogeneities around zero $\delta_i = \xi_i + i\zeta/2$, and define

$$a_i = 3/2 - \epsilon_i = \begin{cases} \ \ 1/2 & (\epsilon_i = 1) \quad \text{for } A\text{-type}, \\ -1/2 & (\epsilon_i = 2) \quad \text{for } D\text{-type}. \end{cases} \tag{6.6}$$

This gives

$$\begin{aligned} \langle E_1^{\epsilon_{\pi(1)}\,\epsilon_{\pi(1)}} \dots E_m^{\epsilon_{\pi(m)}\,\epsilon_{\pi(m)}} \rangle = (-1)^{[\pi]} \int\limits_{\mathcal{C}_D} \mathrm{d}^s\lambda \int\limits_{\widetilde{\mathcal{C}}_A} \mathrm{d}^{m-s}\lambda \ \frac{\det_m \left[\widetilde{\Psi}\left(\lambda_i, \delta_j\right) \right]}{\prod\limits_{i<j} \mathfrak{s}\left(\delta_i, \delta_j\right)} \\ \times \prod_{j>k} \frac{\sinh\left(\delta_k - \lambda_{\pi(j)} + i a_{\pi(j)}\zeta\right) \sinh\left(\delta_j - \lambda_{\pi(k)} - i a_{\pi(k)}\zeta\right)}{\sinh\left(\lambda_{\pi(j)\pi(k)} - i\overline{a}_{\pi(j)\pi(k)}\zeta\right) \sinh\left(\overline{\lambda}_{\pi(j)\pi(k)} - i\overline{a}_{\pi(j)\pi(k)}\zeta\right)} \\ \times \prod_{j,k=1}^{m} \sinh\left(\lambda_j + \delta_k - i a_j \zeta\right) \prod_{j=1}^{m} \frac{\sinh\left(\xi_- + \delta_j\right)}{\sinh\left(\lambda_j + \xi_- - i a_j \zeta\right)} \ . \end{aligned} \tag{6.7}$$

Here

$$\widetilde{\Psi}\left(\lambda, \delta\right) = \Psi\left(\lambda, \delta - i\zeta/2\right) = \frac{\rho\left(\lambda - \delta\right) - \rho\left(\lambda + \delta\right)}{2\sinh 2\delta}, \tag{6.8}$$

$$\widetilde{\mathcal{C}}_A = \begin{cases} \left]-\Lambda\,;\Lambda\right[\cup \Gamma_-\left(\check{\lambda} - \eta\right) & \text{if } -\zeta/2 < \tilde{\xi}_- < 0, \\ \left]-\Lambda\,;\Lambda\right[& \text{otherwise.} \end{cases} \tag{6.9}$$

and $(-1)^{[\pi]}$ is the signature of the permutation.

One can compute the sum over permutations (6.2) just as in the bulk case [52]. It leads to the following integral representation for F_s:

Proposition 6.1 (Bulk-type resummation) *The generating function of the spin correlation function $\langle \mathcal{Q}_m(\kappa) \rangle$ can be expressed as*

$$\langle \mathcal{Q}_m(\kappa) \rangle = \sum_{s=0}^{m} \kappa^s F_s \tag{6.10}$$

with

$$\begin{aligned} F_s = \frac{1}{s!\,(m-s)!} \int\limits_{\mathcal{C}_D} \mathrm{d}^s\lambda \int\limits_{\widetilde{\mathcal{C}}_A} \mathrm{d}^{m-s}\lambda \ \frac{\det_m \left[\widetilde{\Psi}\left(\lambda_i, \delta_j\right) \right]}{\prod\limits_{i<j} \mathfrak{s}\left(\delta_i, \delta_j\right)} \ \prod_{j=1}^{m} \frac{\sinh\left(\xi_- + \delta_j\right)}{\sinh\left(\lambda_j + \xi_- - i a_j \zeta\right)} \\ \times\, \theta_s\left(\{\lambda\}\right) \ Z_m\left(\{\lambda\}, \{\delta\}\right) \prod_{j,k=1}^{m} \sinh\left(\lambda_j + \delta_k - i\zeta a_j\right). \end{aligned} \tag{6.11}$$

Here $Z_m\left(\{\lambda\}, \{\xi\}\right)$ stands for the partition function of the six-vertex model with domain wall boundary conditions:

$$Z_m\left(\{\lambda\}, \{\delta\}\right) = \frac{\prod\limits_{j,k} \mathfrak{s}\left(\lambda_j - \delta_k, i\zeta/2\right)}{\prod\limits_{j<k} \sinh\lambda_{jk} \sinh\delta_{kj}} \ \det_m \left[\frac{1}{\mathfrak{s}\left(\lambda_j - \delta_k, i\zeta/2\right)} \right] \ , \tag{6.12}$$

while

$$\theta_s(\{\lambda\}) = \prod_{j>k} \frac{\sinh \lambda_{jk}}{\mathfrak{s}(\lambda_{jk}, i\overline{a}_{jk}\zeta) \sinh\left(\overline{\lambda}_{jk} - i\overline{a}_{jk}\zeta\right)} \quad . \tag{6.13}$$

Similar representations can be obtained for other correlation functions. Here we give only two important examples: the local density of energy and the $\langle \sigma^+_{m+1}\sigma^-_1 \rangle$ two-point function.

The local density of energy

$$E_m = \langle \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta(\sigma^z_m \sigma^z_{m+1} - 1) \rangle \ , \tag{6.14}$$

can be written as a sum of m terms

$$E_m = \sum_{s=0}^{m-1} \tilde{E}_s, \tag{6.15}$$

each of them containing $m+1$ integrals

$$\begin{aligned}
\tilde{E}_s =& \frac{1}{s!\,(m-1-s)!} \int_{\mathcal{C}_D} \mathrm{d}^s\lambda \int_{\tilde{\mathcal{C}}_A} \mathrm{d}^{m-s-1}\lambda \int_{\mathcal{C}_D} \mathrm{d}\lambda_m \int_{\tilde{\mathcal{C}}_A} \mathrm{d}\lambda_{m+1} \frac{\det_{m+1}\left[\widetilde{\Psi}(\lambda_i, \delta_j)\right]}{\prod_{i<j} \mathfrak{s}(\delta_i, \delta_j)} \\
&\times \theta_s(\{\lambda_1, \ldots, \lambda_{m-1}\})\ Z_m(\{\lambda_1, \ldots, \lambda_{m-1}\}, \{\delta_1, \ldots, \delta_{m-1}\}) \\
&\times \prod_{j=1}^{m+1} \frac{\sinh(\xi_- + \delta_j)}{\sinh(\lambda_j + \xi_- - ia_j\zeta)} \prod_{j=m}^{m+1} \prod_{k=1}^{m-1} \frac{\sinh(\lambda_j - \delta_k - ia_j\zeta)\sinh(\lambda_k - \delta_j + ia_k\zeta)}{\sinh(\lambda_{jk} + i\overline{a}_{jk}\zeta)\sinh(\overline{\lambda}_{jk} - i\overline{a}_{jk}\zeta)} \\
&\times \prod_{j,k=1}^{m+1} \sinh(\lambda_j + \delta_k - ia_j\zeta) \frac{\varphi(\lambda_m, \lambda_{m+1}, \delta_m, \delta_{m+1})}{\sinh(\lambda_{m+1} - \lambda_m)\sinh(\lambda_m + \lambda_{m+1})},
\end{aligned} \tag{6.16}$$

where

$$\begin{aligned}
\varphi(\lambda_m, \lambda_{m+1}, \delta_m, \delta_{m+1}) =& \sinh(\lambda_m - \delta_{m+1} + i\frac{\zeta}{2})\sinh(\lambda_{m+1} - \delta_{m+1} - i\frac{\zeta}{2}) \\
&+ \sinh(\lambda_m - \delta_m - i\frac{\zeta}{2})\sinh(\lambda_{m+1} - \delta_m + i\frac{\zeta}{2}) \\
&- \cos\zeta \sinh(\lambda_m - \delta_{m+1} - i\frac{\zeta}{2})\sinh(\lambda_{m+1} - \delta_m - i\frac{\zeta}{2}) \\
&- \cos\zeta \sinh(\lambda_m - \delta_m + i\frac{\zeta}{2})\sinh(\lambda_{m+1} - \delta_{m+1} + i\frac{\zeta}{2}).
\end{aligned}$$

A similar representation can be obtained for the two-point function $\langle \sigma^+_{m+1}\sigma^-_1 \rangle$, namely:

$$\langle \sigma^+_{m+1}\sigma^-_1 \rangle = \sum_{s=0}^{m-1} G_s, \tag{6.17}$$

and every term contains $m+1$ integrals

$$
\begin{aligned}
G_s =& \frac{(-1)^{m-1}}{s!\,(m-1-s)!} \int\limits_{\mathcal{C}_D} \mathrm{d}^{s+1}\lambda \int\limits_{\widetilde{\mathcal{C}}_A} \mathrm{d}^{m-s}\lambda \frac{\det_{m+1}\left[\widetilde{\Psi}\left(\lambda_i,\delta_j\right)\right]}{\prod\limits_{i<j}\mathfrak{s}\left(\delta_i,\delta_j\right)} \\
&\times\theta_s\left(\{\lambda_2,\ldots,\lambda_m\}\right)\; Z_m\left(\{\lambda_2,\ldots,\lambda_m\},\{\delta_2,\ldots,\delta_m\}\right) \\
&\times\prod_{j=1}^{m+1}\frac{\sinh\left(\xi_-+\delta_j\right)\sinh(\lambda_j-\delta_1-ia_j\zeta)\sinh(\lambda_j-\delta_{m+1}+ia_j\zeta)}{\sinh\left(\lambda_j+\xi_--ia_j\zeta\right)} \\
&\times\prod_{j,k=1}^{m+1}\sinh\left(\lambda_j+\delta_k-ia_j\zeta\right)\prod_{j=1,m+1}\prod_{k=2}^{m}\frac{\sinh(\lambda_j-\delta_k-ia_j\zeta)}{\sinh(\lambda_{jk}+i\overline{a}_{jk}\zeta)\sinh(\overline{\lambda}_{jk}-i\overline{a}_{jk}\zeta)} \\
&\times\frac{\sinh(\lambda_1-\delta_1-i\frac{\zeta}{2})\sinh(\lambda_{m+1}-\delta_1+i\frac{\zeta}{2})}{\sinh(\lambda_{m+1}-\lambda_1)\sinh(\lambda_1+\lambda_{m+1})},
\end{aligned}
\tag{6.18}
$$

6.2 Boundary type resumations

It is important to note that the function Z_m appearing in the representations (6.11), (6.16) and (6.18) is the bulk partition function represented in terms of the Izergin determinant [71]. In the boundary case one can symmetrize the integrand even further by writing it in a form invariant under the reversal of the parameters λ and finally rewrite the result in terms of the boundary partition function and the Tsuchiya determinant. The integration contours in (6.11) are not invariant under the transformation $\lambda \to -\lambda$. We thus deform the contours until we obtain a reversal invariant contour. As we do not cross any pole of the integrand, the result remains unchanged. Actually we can even pick the contours so as to integrate only over one contour $\mathcal{C}$, although this is not necessary. This contour $\mathcal{C}$ is defined as follows according to the value of the boundary field h_-:

$\zeta/2 <\mid \tilde{\xi}_- \mid< \pi/2$	$\mathcal{C}=\mathbb{R}$
$\zeta/2 > \tilde{\xi}_- > 0$	$\mathcal{C}=\mathbb{R}\,\bigcup\,\Gamma_+(\check{\lambda})\bigcup\,\Gamma_-(-\check{\lambda})$
$-\zeta/2 < \tilde{\xi}_- < 0$	$\mathcal{C}=\mathbb{R}\,\bigcup\,\Gamma_+(i\zeta+\check{\lambda})\bigcup\,\Gamma_-(-i\zeta-\check{\lambda})$

(6.19)

We extract the totally even part of the integrand appearing in (6.7) according to

$$
\int\limits_{\mathcal{C}}\mathrm{d}x f\left(x\right)=\frac{1}{2}\sum_{\sigma=\pm}\int\limits_{\mathcal{C}}\mathrm{d}x f\left(x^{\sigma}\right)\;,\qquad x^{\sigma}=\sigma\,x\;. \tag{6.20}
$$

We get

$$
\begin{aligned}
F_s=\frac{1}{s!(m-s)!2^m}\int\limits_{\mathcal{C}}\mathrm{d}^m\lambda\;\frac{\det_m\left[\widetilde{\Psi}\left(\lambda_i,\delta_j\right)\right]}{\prod\limits_{i<j}\mathfrak{s}\left(\delta_i,\delta_j\right)}\;\prod_{j=1}^{m}\frac{\sinh\left(\xi_-+\delta_j\right)}{\sinh\left(\lambda_j,\xi_--ia_j\zeta\right)} \\
\times\,\Theta_s\left(\{\lambda\}\right)\;H_s\left(\{\lambda\},\{\delta\}\right)\quad.
\end{aligned}
\tag{6.21}
$$

Here

$$\Theta_s(\{\lambda\}) = \prod_{j>k} \frac{\mathfrak{s}(\lambda_j, \lambda_k)}{\mathfrak{s}(\lambda_{jk}, i\overline{a}_{jk}\zeta)\,\mathfrak{s}\left(\overline{\lambda}_{jk}, i\overline{a}_{jk}\zeta\right)} \quad , \tag{6.22}$$

and the sums over negations have been absorbed into $H_s(\{\lambda\}, \{\delta\})$:

$$H_s(\{\lambda\}, \{\delta\}) = \sum_{\sigma_i=\pm} \prod_{j=1}^{m} \left[\sigma_j \sinh\left(\lambda_j^\sigma - \xi_- - ia_j\zeta\right)\right] \prod_{j>k} \frac{\sinh\left(\overline{\lambda}_{jk}^\sigma + i\overline{a}_{jk}\zeta\right)}{\sinh\left(\overline{\lambda}_{jk}^\sigma\right)} \\ \times \prod_{j,k}^{m} \sinh\left(\lambda_j^\sigma + \delta_k - ia_j\zeta\right) \; \mathrm{Z}_m(\{\lambda^\sigma\}, \{\delta\}) . \tag{6.23}$$

Equation (6.23) implies that $H_s(\{\lambda\}, \{\delta\})$ is a symmetric function of the parameters λ and of the parameters δ. Moreover, $\mathrm{e}^{2(m-1)\lambda_j}\ H_s(\{\lambda\}, \{\delta\})$ is a polynomial in each of the $\mathrm{e}^{2\lambda_j}$ variables of degree $2(m-1)$. Finally, it is a matter of straightforward computations to check that H_s satisfies the reduction properties:

$$H_s\,|_{\lambda_1=\pm(\delta_1-ia_1\zeta)} \left(\{\lambda_i\}_{i=1}^m ; \{\delta_k\}_{k=1}^m\right) = \pm H_s\left(\{\lambda_i\}_{i=2}^m ; \{\delta_k\}_{k=2}^m\right) \\ \times \sinh\left(2\left(\delta_1 - ia_1\zeta\right)\right) \sinh\left(\delta_1 - \xi_-\right) \prod_{j=2}^{m} \mathfrak{s}(\lambda_j, \delta_1 + ia_1\zeta)\, \mathfrak{s}(\delta_1 - 2ia_1\zeta, \delta_k) . \tag{6.24}$$

These are the reduction properties of $\mathcal{Z}_m(\{\lambda\}, \{\delta\})$, the partition function of the six-vertex model with reflecting ends [72]. Supplementing this result with the equality of the two functions at $m = 1$, we obtain that $H_s(\{\lambda\}; \{\delta\})$ is s-independent and equal to $\mathcal{Z}_m(\{\lambda\}, \{\delta\})$. Hence, we have the following result:

Proposition 6.2 (Boundary-type resummation) *The generating function of the spin correlation function $\langle \mathcal{Q}_m(\kappa) \rangle$ can be expressed as*

$$\langle \mathcal{Q}_m(\kappa) \rangle = \sum_{s=0}^{m} \kappa^s F_s \tag{6.25}$$

with

$$F_s = \frac{1}{s!\,(m-s)!2^m} \int_{\mathcal{C}} \mathrm{d}^m\lambda \; \frac{\det_m\left[\widetilde{\Psi}(\lambda_i, \delta_j)\right]}{\prod_{i<j} \mathfrak{s}(\delta_i, \delta_j)} \; \prod_{j=1}^{m} \frac{\sinh(\xi_- + \delta_j)}{\sinh(\lambda_j, \xi_- - ia_j\zeta)} \\ \times \Theta_s(\{\lambda\}) \;\; \mathcal{Z}_m(\{\lambda\}, \{\delta\}) \quad . \tag{6.26}$$

Here

$$\Theta_s(\{\lambda\}) = \prod_{j>k} \frac{\mathfrak{s}(\lambda_j, \lambda_k)}{\mathfrak{s}(\lambda_{jk}, i\overline{a}_{jk}\zeta)\,\mathfrak{s}\left(\overline{\lambda}_{jk}, i\overline{a}_{jk}\zeta\right)} \quad , \tag{6.27}$$

and

$$\mathcal{Z}_m\left(\{\lambda\},\{\delta\}\right)=\frac{\prod\limits_{j,k=1}^{m}\left[\mathfrak{s}\left(\lambda_j,\delta_k+i\zeta/2\right)\,\mathfrak{s}\left(\lambda_j,\delta_k-i\zeta/2\right)\right]}{\prod\limits_{i<j}\left[\mathfrak{s}\left(\lambda_i,\lambda_j\right)\,\mathfrak{s}\left(\delta_j,\delta_i\right)\right]}$$
$$\times\prod_{j=1}^{m}\left[\sinh 2\lambda_j\,\sinh\left(\delta_j-\xi_-\right)\right]\,\det\nolimits_m\left[\frac{1}{\mathfrak{s}\left(\lambda_j,\delta_k+i\zeta/2\right)\mathfrak{s}\left(\lambda_j,\delta_k-i\zeta/2\right)}\right].\qquad(6.28)$$

Let us finally mention that the sum over s in (6.2) can be rewritten as a single integral over auxiliary z variables:

$$\langle\mathcal{Q}_m\left(\kappa\right)\rangle=\frac{1}{m!\,2^m}\int\limits_{\mathcal{C}}\mathrm{d}^m\lambda\oint\limits_{\bigcup\limits_{\epsilon=\pm}\Gamma_+\left(\epsilon i\frac{\zeta}{2}\right)}\frac{\mathrm{d}^m z}{\left(2i\pi\right)^m}\,\frac{\mathcal{Z}_m\left(\{\lambda\},\{\delta\}\right)\,\det_m\left[\widetilde{\Psi}\left(\lambda_i,\delta_j\right)\right]}{\prod\limits_{j>k}^{m}\left[\mathfrak{s}\left(\lambda_j,\lambda_k\right)\mathfrak{s}\left(\lambda_{jk},i\zeta\right)\mathfrak{s}\left(\overline{\lambda}_{jk},i\zeta\right)\right]}$$
$$\times\prod_{k>l}^{m}\frac{\mathfrak{s}\left(\lambda_{kl},z_{kl}\right)\mathfrak{s}\left(\overline{\lambda}_{kl},z_{kl}\right)}{\mathfrak{s}\left(\delta_l,\delta_k\right)}\,\prod_{p=1}^{m}\frac{\varphi\left(z_p\right)\sinh\left(\xi_-+\xi_p\right)}{\mathfrak{s}\left(\lambda_p,\xi_-+z_p\right)},\qquad(6.29)$$

where

$$\varphi\left(z\right)=\frac{\sinh 2z\;\kappa^{-i(z+i\zeta/2)/\zeta}}{\mathfrak{s}\left(z,i\zeta/2\right)}.\qquad(6.30)$$

This boundary-type resummation yields an integrand not only symmetric in $\{\lambda\}$ but also invariant under a reversal of any integration variable λ. These properties allow to compute completely the so called emptiness formation probability at $\Delta=1/2$ and for vanishing boundary magnetic fields. It also allows to obtain the leading asymptotics of this quantity at the free fermion point [75].

7 The free fermion point

7.1 Local magnetization at distance m

The first (and the most important) application of the re-summation methods given above is the magnetization profile. This one-point function

$$\langle\sigma_m^z\rangle=1-2D_m\partial_\kappa\langle\mathcal{Q}_m\left(\kappa\right)\rangle\,|_{\kappa=1},$$

can be computed at the free fermion point by using the two different types of re-summations for the generation function. We give here both derivations.

7.1.1 First method

In the free fermion point, the n^{th} term of the series (5.1) behaves as $(\kappa-1)^n$. Thus, after taking the κ derivative and sending κ to 1, only the $n=1$ term survives.

At $\zeta = \pi/2$, we have

$$\langle \mathcal{Q}_m(\kappa) \rangle = \sum_{n=0}^{m} \frac{(\kappa-1)^n}{(2\pi)^{2n}(n!)^2} \int_{\mathcal{C}_D} \mathrm{d}^n\lambda \oint_{\Gamma_+(\{\xi\})} \mathrm{d}^n z \prod_{a=1}^{m}\prod_{b=1}^{n} \frac{\tanh(\lambda_b - \xi_a)}{\tanh(z_b - \xi_a)}$$
$$\times \prod_{j=1}^{n} \left\{ \frac{\sinh(z_j + \xi_- + i\pi/4)}{\sinh(\lambda_j + \xi_- + i\pi/4)} \sinh 2\lambda_j \right\}$$
$$\times \det_n \left[\frac{1}{\sinh(\lambda_j - z_k)} \right] \det_n \left[\frac{1}{\mathfrak{s}(\lambda_j, z_k)} \right]. \tag{7.1}$$

Thus [6], for $m \geq 2$,

$$\langle \sigma_m^z \rangle = \frac{(-1)^m}{\pi} \int_{\mathcal{C}_D} \mathrm{d}\lambda \, \frac{\sinh(\lambda - \xi_- - i\pi/4)}{\sinh(\lambda + \xi_- + i\pi/4)} \frac{[\tanh(\lambda + i\pi/4)]^{2(m-1)}}{\cosh^2(\lambda + i\pi/4)}. \tag{7.2}$$

Computing, if it exists, the residue at $\check{\lambda}$ we get, for $m \geq 2$,

$$\langle \sigma_m^z \rangle = -2\Theta(h_- - 1) \frac{h_-^2 - 1}{h_-^{2m}}$$
$$+ \frac{(-1)^m}{\pi} \int_{\mathbb{R}} \mathrm{d}\lambda \, \frac{h_- + i\tanh(\lambda + i\pi/4)}{1 + ih_- \tanh(\lambda + i\pi/4)} \, \frac{[\tanh(\lambda + i\pi/4)]^{2(m-1)}}{\cosh^2(\lambda + i\pi/4)}, \tag{7.3}$$

where $\Theta(x)$ is the Heaviside step function. The standard $\Delta = 0$ change of variables,

$$\mathrm{e}^{ip} = -\tanh(\lambda - i\pi/4) \ , \tag{7.4}$$

yields

$$\langle \sigma_m^z \rangle = -2\Theta(h_- - 1) \frac{h_-^2 - 1}{h_-^{2m}} + \frac{(-1)^m}{\pi} \int_0^{\pi} \mathrm{d}p \, \mathrm{e}^{-2i(m-1)p} \, \frac{\mathrm{e}^{-ip} + ih_-}{\mathrm{e}^{ip} - ih_-}. \tag{7.5}$$

Thus $\langle \sigma_m^z \rangle$ displays Friedel type oscillations induced by the boundary. Moreover it decays as $1/m$ when $m \to +\infty$:

$$\langle \sigma_m^z \rangle = \frac{2(-1)^m}{\pi m} \frac{h_-}{h_-^2 + 1} + \mathrm{O}\left(\frac{1}{m^2}\right), \qquad m >> 1. \tag{7.6}$$

Here we recover the results of [69], since we have $h_- = \sqrt{2}\alpha_-$ in Bilstein's notations. When $\mid h_- \mid \to \infty$ we conclude from (7.5) that the first site is totally decoupled from the others as $\langle \sigma_m^z \rangle_{m \geq 2}$ goes to its bulk average value 0. Actually in this limit the model is in correspondence with a Kondo model with a spin 1/2 impurity [29].

[6] We do not give $\langle \sigma_1^z \rangle$ as it corresponds to $\partial_\kappa \langle \mathcal{Q}_1(\kappa) \rangle$ without taking the lattice derivative.

But in this case the impurity is completely screened, and the overall magnetization in zero.

We also recover from (7.5), just as expected from the spin reversal symmetry, that $\langle \sigma^z_m \rangle = 0$ when the boundary field vanishes. Actually this observation holds for all Δ as inferred from the structure of the monodromy matrix (2.6) on the first site. When one sets $\xi_1 = \eta/2$ and $\xi_- = 0$ then it acts as a diagonal matrix on the first site, a sign of the claimed decoupling.

7.1.2 Second method

Starting from the re-summation formula (6.11) of Proposition 6.1, we implement the simplification due to $\zeta = \pi/2$. If we perform the change of variables

$$\mathrm{e}^{ip} = -\tanh\left(\lambda - i\pi/4\right) \tag{7.7}$$

in (6.11) at $\zeta = \pi/2$ then we arrive at

$$\begin{aligned} F_s = {} & \frac{(2i)^{\frac{m(m-1)}{2}}}{s!\,(m-s)!} \prod_{j=1}^{s} \int_{\mathcal{C}_D} \frac{dp_j}{2\pi} \left[\frac{\mathrm{e}^{ip_j} + \mathrm{e}^{-ip_j}}{\mathrm{e}^{ip_j} - ih_-}\right] \prod_{j=s+1}^{m} \int_{\overline{\mathcal{C}}_A} \frac{dp_j}{2\pi} \left[\frac{\mathrm{e}^{ip_j} + \mathrm{e}^{-ip_j}}{\mathrm{e}^{-ip_j} - ih_-}\right] \\ & \times \prod_{k=1}^{s} \prod_{j=s+1}^{m} \left[\left(\mathrm{e}^{-ip_k} - \mathrm{e}^{ip_j}\right) \left(\sin p_j + \sin p_k\right) \right] \prod_{\substack{j,k=1 \\ j>k}}^{s} \left[\left(\mathrm{e}^{-ip_j} - \mathrm{e}^{-ip_k}\right) \left(\sin p_j - \sin p_k\right) \right] \\ & \times \prod_{\substack{j,k=s+1 \\ j>k}}^{m} \left[\left(\mathrm{e}^{ip_j} - \mathrm{e}^{ip_k}\right) \left(\sin p_k - \sin p_j\right) \right]. \end{aligned} \tag{7.8}$$

The contours of integration are

$$\overline{\mathcal{C}}_A = \left]\,0\,;\pi\,\right[, \quad h_- \geq -1, \qquad \overline{\mathcal{C}}_A = \left]\,0\,;\pi\,\right[\cup \Gamma_-\left(\mathrm{e}^{-ip} = -ih_-\right), \quad h_- < -1, \tag{7.9}$$

$$\mathcal{C}_A = \left]\,-\pi\,;0\,\right[, \quad h_- \geq -1, \qquad \mathcal{C}_A = \left]\,-\pi\,;0\,\right[\cup \Gamma_+\left(\mathrm{e}^{ip} = -ih_-\right), \quad h_- < -1, \tag{7.10}$$

$$\mathcal{C}_D = \left]\,0\,;\pi\,\right[, \quad h_- \leq 1, \qquad \mathcal{C}_D = \left]\,0\,;\pi\,\right[\cup \Gamma_+\left(\mathrm{e}^{ip} = -ih_-\right), \quad h_- > 1. \tag{7.11}$$

Once we introduce the function

$$\theta_\kappa\left(p\right) = \begin{cases} \kappa & p \in \mathcal{C}_D, \\ 1 & p \in \mathcal{C}_A, \end{cases} \tag{7.12}$$

we can re-sum the terms F_s into a single m-fold integral for $\langle Q_m\left(\kappa\right)\rangle$:

$$\begin{aligned} \langle \mathcal{Q}_m\left(\kappa\right)\rangle \quad = \quad & \frac{(2i)^{\frac{m(m-1)}{2}}}{m!} \int_{\mathcal{C}_A \cup \mathcal{C}_D} \prod_{j=1}^{m} \left[\frac{\mathrm{d}p_j}{2\pi}\, \theta_\kappa\left(p_j\right) \frac{\mathrm{e}^{ip_j} + \mathrm{e}^{-ip_j}}{\mathrm{e}^{ip_j} - ih_-}\right] \\ & \times \prod_{j>k\geq 1}^{m} \left(\mathrm{e}^{-ip_j} - \mathrm{e}^{-ip_k}\right) \left(\sin p_j - \sin p_k\right). \end{aligned} \tag{7.13}$$

One can then express the generating function as a single determinant

$$\begin{aligned} \langle \mathcal{Q}_m(\kappa) \rangle &= \det_m [U(\kappa)], && (7.14)\\ U_{jk}(\kappa) &= \frac{1}{2\pi} \int\limits_{\mathcal{C}_A \cup \mathcal{C}_D} \mathrm{d}p\ \theta_\kappa(p)\ \mathrm{e}^{-ip(j-1)} \frac{\mathrm{e}^{ipk} - (-1)^k \mathrm{e}^{-ipk}}{\mathrm{e}^{ip} - ih_-}. && (7.15) \end{aligned}$$

To simplify this result we add to each row of $U(\kappa)$ the next one multiplied by ih_-:

$$\begin{aligned} &\langle \mathcal{Q}_m(\kappa) \rangle = \det_m \widetilde{U}^{(m)}(\kappa),\\ &\widetilde{U}^{(m)}_{jk}(\kappa) = \frac{1}{2\pi} \int\limits_{\mathcal{C}_A \cup \mathcal{C}_D} \mathrm{d}p\ \theta_\kappa(p) \left(\mathrm{e}^{ip(k-j)} - (-1)^k \mathrm{e}^{-ip(j+k)} \right), \quad j < m,\\ &\widetilde{U}^{(m)}_{mk}(\kappa) = U_{mk}(\kappa). \end{aligned}$$

It is easy to see that $Q_m(1) = 1$. Computing the first derivative of the generating function one recovers the result already obtained from the series (7.5):

$$\begin{aligned} \langle \sigma^z_m \rangle &= \frac{(-1)^m}{\pi} \int\limits_{\mathcal{C}_D} \mathrm{d}p\ \mathrm{e}^{-2ip(m-1)} \frac{\mathrm{e}^{-ip} + ih_-}{\mathrm{e}^{ip} - ih_-}.\\ &= -2 \frac{h_-^2 - 1}{h_-^{2m}} \Theta(h_- - 1) + \frac{(-1)^m}{\pi} \int\limits_0^\pi \mathrm{d}p\ \mathrm{e}^{-2ip(m-1)} \frac{\mathrm{e}^{-ip} + ih_-}{\mathrm{e}^{ip} - ih_-}. && (7.16) \end{aligned}$$

7.2 Local density of energy

The local density of energy is another interesting quantity [61] that one can evaluate for the XX0 chain:

$$E_m = \langle \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} \rangle\ . \tag{7.17}$$

Starting from (6.16) and using the same technique as in the previous sub-section one easily obtains the following representation for the density of energy:

$$E_m = -2i \int\limits_{\mathcal{C}_D} \frac{\mathrm{d}p}{2\pi} \int\limits_{\mathcal{C}_A} \frac{\mathrm{d}q}{2\pi} \left(\mathrm{e}^{i(p+q)} + 1 \right) \det_{m+1} [V(p,q)]\ . \tag{7.18}$$

The entries of $V\,(p,q)$ read

$$V_{jk}(p,q) = \frac{1}{2\pi} \int\limits_{\mathcal{C}_A \cup \mathcal{C}_D} \mathrm{d}p' \left(e^{ip'(k-j)} - (-1)^k e^{-ip'(j+k)} \right) = \delta_{jk}, \qquad j < m-1,$$

$$V_{m-1k}(p,q) = \frac{1}{2\pi} \int\limits_{\mathcal{C}_A \cup \mathcal{C}_D} \mathrm{d}p' \mathrm{e}^{-ip'(m-2)} \frac{\mathrm{e}^{ip'k} - (-1)^k \mathrm{e}^{-ip'k}}{\mathrm{e}^{ip'} - ih_-} = \left(\frac{h_-}{i} \right)^{k-m+1} \Theta\,(k-m+1)\,,$$

$$V_{mk}(p,q) = \mathrm{e}^{-ipm} \frac{\mathrm{e}^{ipk} - (-1)^k \mathrm{e}^{-ipk}}{\mathrm{e}^{ip} - ih_-},$$

$$V_{m+1k}(p,q) = \mathrm{e}^{-iqm} \frac{\mathrm{e}^{iqk} - (-1)^k \mathrm{e}^{-iqk}}{\mathrm{e}^{iq} - ih_-}. \tag{7.19}$$

Finally, the integrals over $\mathcal{C}_A$ can be represented as

$$\int\limits_{\mathcal{C}_A} = \int\limits_{\mathcal{C}_A \cup \mathcal{C}_D} - \int\limits_{\mathcal{C}_D} . \tag{7.20}$$

Accordingly, E_m reduces to a sum of two 3×3 determinants:

$$\begin{aligned} E_m \;\; = \;\; & -2i \begin{vmatrix} 1 & ih_- & -h_-^2 \\ F(m-1,m-1) & F(m-1,m) & F(m-1,m+1) \\ 0 & 1 & ih_- \end{vmatrix} \\ & -2i \begin{vmatrix} 1 & ih_- & -h_-^2 \\ F(m,m-1) & F(m,m) & F(m,m+1) \\ 0 & 0 & 1 \end{vmatrix} , \end{aligned} \tag{7.21}$$

where

$$F\,(j,k) = \frac{1}{2\pi} \int\limits_{\mathcal{C}_D} \mathrm{d}p \;\mathrm{e}^{-ipj} \;\frac{\mathrm{e}^{ipk} - (-1)^k \mathrm{e}^{-ipk}}{\mathrm{e}^{ip} - ih_-} \,. \tag{7.22}$$

The computation of these determinants yields

$$E_m = -2i\,(F(m,m) - F(m-1,m+1)) - 2h_-\,(F(m,m-1) - F(m-1,m))\;, \tag{7.23}$$

or, more explicitly,

$$E_m = -\frac{4}{\pi} + \frac{2i}{\pi}\,(-1)^m \int\limits_{\mathcal{C}_D} \mathrm{d}p \;\mathrm{e}^{-ip(2m-1)} \;\frac{\mathrm{e}^{-ip} + ih_-}{\mathrm{e}^{ip} - ih_-} \,. \tag{7.24}$$

The constant term reproduces the bulk result. The influence of the boundary appears in the oscillating term. In the $m \to \infty$ limit the local density of energy behaves as

$$E_m = -\frac{4}{\pi} + \frac{2}{\pi m}\,(-1)^m \,\frac{1-h_-^2}{1+h_-^2} + \mathrm{O}\left(\frac{1}{m^2}\right). \tag{7.25}$$

7.2.1 Two-point function $\langle \sigma_{m+1}^+ \sigma_1^- \rangle$

The preceding method can be successfully applied to compute other types of two-point functions (like boundary-bulk two point functions), here we give the example of $\langle \sigma_{m+1}^+ \sigma_1^- \rangle$.

In the free fermion point, after the usual change of variables and some straightforward but tedious calculations, one obtains from (6.18) a simple determinant formula for this object.

$$\langle \sigma_{m+1}^+ \sigma_1^- \rangle = - i \det_{m+1} V^{+-}, \tag{7.26}$$

$$V_{jk}^{+-} = \frac{1}{2\pi i} \left(\int_0^\pi - \int_\pi^{2\pi} \right) dp \left(e^{ip(k-j-1)} - (-1)^k e^{-ip(j+1+k)} \right), \quad j \leq m-1,$$

$$V_{mk}^{+-} = \frac{1}{2\pi} \int_{\mathcal{C}_D} \mathrm{d}p \, e^{-ipm} \frac{e^{ipk} - (-1)^k e^{-ipk}}{e^{ip} - ih_-},$$

$$V_{m+1k}^{+-} = \frac{1}{2\pi} \int_{\mathcal{C}_A} dp \, e^{-ipm} \frac{e^{ipk} - (-1)^k e^{-ipk}}{e^{ip} - ih_-} \tag{7.27}$$

Computing the integrals in the first $m-1$ rows and using the fact that sum of the last two rows is $\delta_{k,m+1}$ we reduce this representation to a determinant of a $m \times m$ matrix

$$\langle \sigma_{m+1}^+ \sigma_1^- \rangle = \frac{i(-1)^m}{2\pi^m} \det_m \tilde{V}^{+-}, \tag{7.28}$$

$$\tilde{V}_{jk}^{+-} = (1 + (-1)^{j-k}) \frac{(j+1)(1-(-1)^k) + k(1+(-1)^k)}{(j+1)^2 - k^2}, \quad j \leq m-1,$$

$$\tilde{V}_{mk}^{+-} = \int_{\mathcal{C}_D} \mathrm{d}p \, e^{-ipm} \frac{e^{ipk} - (-1)^k e^{-ipk}}{e^{ip} - ih_-}, \tag{7.29}$$

This determinant can be computed for any value of m. However the result is quite different for m odd or even. The details of the computation are given in Appendix

B, here we give only the final result for the two point function

$$\langle\sigma_{2a}^{+}\sigma_{1}^{-}\rangle = -\, i\frac{2^{2a-1}}{\pi^2}\left(\prod_{j=1}^{2a-1}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2\frac{\Gamma(a-\frac{1}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}$$

$$\times\int\limits_{\mathcal{C}_D}\mathrm{d}p\,P_{2a-1}\Big(\cos p\Big)\,\frac{e^{-ip(2a-1)}}{e^{ip}-ih_-} \tag{7.30}$$

$$\langle\sigma_{2a+1}^{+}\sigma_{1}^{-}\rangle = -\,\frac{2^{2a}}{\pi^2}\left(\prod_{j=1}^{2a}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2\frac{\Gamma(a+\frac{3}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}$$

$$\times\int\limits_{0}^{\pi}dq\,\cos q\int\limits_{\mathcal{C}_D}\mathrm{d}p\,P_{2a}\Big(\cos(q-p)\Big)\,\frac{e^{-2ipa}}{e^{ip}-ih_-}, \tag{7.31}$$

where $P_m(x)$ are Legendre polynomials.

Asymptotic analysis of these expression yields the following leading behavior of $\langle\sigma_{m+1}^{+}\sigma_{1}^{-}\rangle$

$$\langle\sigma_{m+1}^{+}\sigma_{1}^{-}\rangle = (-1)^m A(h_-)\,m^{-\frac{3}{4}}\left(1+O(\frac{1}{\sqrt{m}})\right) \tag{7.32}$$

$$A(h_-) = \sqrt{\frac{2}{\pi(1+h_-^2)}}\,\exp\left\{\frac{1}{4}\int_0^\infty\frac{dt}{t}\left[e^{-4t}-\frac{1}{\cosh^2 t}\right]\right\}. \tag{7.33}$$

8 Conclusion

In this article we have obtained different types of physical correlation functions of the open XXZ chain from re-summations of the multiple integrals derived in [4] for the elementary blocks. At the free-fermion point, we were able to use these representations to derive explicit results such as the formula for the density of energy profiles, a quantity arising in the study of quantum entanglement in spin chains [61].

Just as in the bulk case, the question concerning the asymptotic behavior of the correlation functions outside of the free-fermion point naturally arises. The problem is of the same order of difficulty as in the bulk model. Indeed, the multiple integrals differ from their bulk counterparts only by factors due to the $\mathbb{Z}_2$ symmetry $\lambda \to -\lambda$ and the presence of boundary fields.

One could also wonder if it would be possible to tell something about the dynamical or temperature correlation functions. It seems that this generalization is highly non-trivial.

Finally, we would like to stress that our expressions also simplify at other particular points such as $\Delta = 1/2$. For instance, when $\Delta = 1/2$, one can already compute completely the so-called emptiness formation probability when $h_- = 0$ [75].

Appendices

A Asymptotic of the two-point function $\langle \sigma^+_{m+1}\sigma^-_1\rangle$

In the last section we obtained a determinant representation (7.29) for the two-point function $\langle \sigma^+_{m+1}\sigma^-_1\rangle$. This determinant can be computed for any value of m. However the results are quite different for m odd or even.

If m is odd: $m = 2a-1$, the result can be written in the following form

$$\begin{aligned}\det{}_m \tilde{V}^{+-} =& 2^{m+1}\pi^{m-3}\left(\prod_{j=1}^{m}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2 \frac{\Gamma(a-\frac{1}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}\\ &\times\sum_{b=1}^{a}\frac{\Gamma(a-b+\frac{1}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b)}\\ &\times\int_0^\pi dp\, e^{-ip(2a-1)}\frac{e^{ip(2b-1)}+e^{-ip(2b-1)}}{e^{ip}-ih_-}\end{aligned}\tag{A.1}$$

If m is even: $m = 2a$, the result is quite similar but there is a very important difference:

$$\begin{aligned}\det{}_m \tilde{V}^{+-} =& 2^{m+1}\pi^{m-3}\left(\prod_{j=1}^{m}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2 \frac{\Gamma(a+\frac{3}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}\\ &\times\sum_{b=1}^{a}\frac{b}{(b+\frac{1}{2})(b-\frac{1}{2})}\frac{\Gamma(a-b+\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b+1)}\\ &\times\int_0^\pi dp\, e^{-2ipa}\frac{e^{2ipb}-e^{-2ipb}}{e^{ip}-ih_-}\end{aligned}\tag{A.2}$$

Asymptotic analysis of the prefactors in (A.1) and (A.2) is rather simple, namely:

$$\begin{aligned}&\left(\prod_{j=1}^{2a-1}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2 \frac{\Gamma(a-\frac{1}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}\\ &\qquad=\frac{\sqrt{2}\pi}{2^m}m^{-\frac{1}{4}}\exp\left\{\frac{1}{4}\int_0^\infty\frac{dt}{t}\left[e^{-4t}-\frac{1}{\cosh^2 t}\right]\right\}\left(1+O(\frac{1}{m})\right)\end{aligned}\tag{A.3}$$

$$\begin{aligned}&\left(\prod_{j=1}^{2a}\frac{\Gamma(j)}{\Gamma(j+\frac{1}{2})}\right)^2 \frac{\Gamma(a+\frac{3}{2})\Gamma^3(a+\frac{1}{2})}{\Gamma(a)\Gamma(a+1)}\\ &\qquad=\frac{\sqrt{2}\pi}{2^{m+1}}m^{\frac{3}{4}}\exp\left\{\frac{1}{4}\int_0^\infty\frac{dt}{t}\left[e^{-4t}-\frac{1}{\cosh^2 t}\right]\right\}\left(1+O(\frac{1}{m})\right)\end{aligned}\tag{A.4}$$

For m odd the sum in (A.1) can be rewritten as follows:

$$\sum_{b=1}^{a} \frac{\Gamma(a-b+\frac{1}{2})\Gamma(a+b-\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b)} \left(e^{-2ip(a-b)}+e^{-2ip(a+b-1)}\right)$$
$$= \sum_{l=0}^{2a-1} \frac{\Gamma(l+\frac{1}{2})\Gamma(2a-l-\frac{1}{2})}{\Gamma(l+1)\Gamma(2a-l)} e^{-2ipl}, \qquad \text{(A.5)}$$

and can be represented in terms of the Legendre polynomials $P_m(\cos p)$

$$\sum_{l=0}^{2a-1} \frac{\Gamma(l+\frac{1}{2})\Gamma(2a-l-\frac{1}{2})}{\Gamma(l+1)\Gamma(2a-l)} e^{-2ipl} = \frac{\Gamma(2a-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2a)}\, {}_2F_1(\frac{1}{2}, 1-2a; \frac{3}{2}-2a; e^{-2ip})$$
$$= \pi e^{-ipm} P_m(\cos p) \qquad \text{(A.6)}$$

Using Laplace asymptotic formula,

$$P_m(\cos p) = \left(\frac{2}{\pi m \sin p}\right)^{\frac{1}{2}} \cos\left[p\left(m+\frac{1}{2}\right)-\frac{\pi}{4}\right] + O\left(\frac{1}{m^{\frac{3}{2}}}\right), \qquad \text{(A.7)}$$

for the remaining integral we obtain the following leading term

$$\int_0^{\pi} dp\, P_m(\cos p) \frac{e^{-ipm}}{e^{ip}-ih_-} = -i\sqrt{\frac{\pi}{m(1+h_-^2)}} \left(1+O(\frac{1}{\sqrt{m}})\right) \qquad \text{(A.8)}$$

Assembling all the contributions we obtain the following leading term for the two-point function (for m odd):

$$\langle \sigma^+_{m+1}\sigma^-_1 \rangle = (-1)^m \sqrt{\frac{2}{\pi(1+h_-^2)}} \exp\left\{\frac{1}{4}\int_0^{\infty} \frac{dt}{t}\left[e^{-4t}-\frac{1}{\cosh^2 t}\right]\right\} m^{-\frac{3}{4}} \left(1+O(\frac{1}{\sqrt{m}})\right) \qquad \text{(A.9)}$$

The same result holds for m even, but the derivation is a little bit more tricky. The sum in (A.2) can be once again rewritten in a more simple way

$$\sum_{b=1}^{a} \frac{b}{(b+\frac{1}{2})(b-\frac{1}{2})} \frac{\Gamma(a-b+\frac{1}{2})(h_-)\Gamma(a+b+\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b+1)} \int_0^{\pi} dp\, e^{-2ipa} \frac{e^{2ipb}-e^{-2ipb}}{e^{ip}-ih_-}$$
$$= \frac{1}{2}\sum_{b=-a}^{a} \left(\frac{1}{b+\frac{1}{2}}+\frac{1}{b-\frac{1}{2}}\right) \frac{\Gamma(a-b+\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b+1)} \int_0^{\pi} dp\, \frac{e^{2ip(b-a)}}{e^{ip}-ih_-}$$
$$= i\sum_{b=-a}^{a} \frac{\Gamma(a-b+\frac{1}{2})\Gamma(a+b+\frac{1}{2})}{\Gamma(a-b+1)\Gamma(a+b+1)} \int_0^{\pi} dq\, e^{-2iqb}\cos q \int_0^{\pi} dp\, \frac{e^{2ip(b-a)}}{e^{ip}-ih_-}$$
$$= i\pi \int_0^{\pi} dq\, \cos q \int_0^{\pi} dp\, P_m(\cos(q-p)) \frac{e^{-imp}}{e^{ip}-ih_-} \qquad \text{(A.10)}$$

where we introduced an additional integral to be able to express the result once again in terms of the Legendre polynomials. Asymptotic analysis of these integrals gives

$$i\pi \int_0^\pi dq \, \cos q \int_0^\pi dp \, P_m(\cos(q-p)) \frac{e^{-imp}}{e^{ip} - ih_-} = -\left(\frac{\pi}{m}\right)^{\frac{3}{2}} \frac{2i}{\sqrt{1+h_-^2}} \left(1 + O(\frac{1}{\sqrt{m}})\right), \tag{A.11}$$

and it leads once again to the same leading term (A.9) for the two-point function.

Acknowledgments

J.M. M., N. S. and V. T. are supported by CNRS. N. K., K.K. K., J.M. M. and V. T. are supported by the ANR programm GIMP ANR-05-BLAN-0029-01. N. K., G. N. and V. T. are supported by the ANR programm MIB-05 JC05-52749. N. S. is supported by the French-Russian Exchange Program, the Program of RAS Mathematical Methods of the Nonlinear Dynamics, RFBR-05-01-00498, Scientific Schools 672.2006.1. N. K., G. N. and N. S. would like to thank the Theoretical Physics group of the Laboratory of Physics at ENS Lyon for hospitality, which makes this collaboration possible.

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